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PRINCIPLES OF COMMUNICATIONS

Systems, Modulation, and Noise

Second Edition

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Solutions Manual

PRINCIPLES OF COMMUNICATIONS
Systems, Modulation, and Noise
Second Edition

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and

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PART II

PROBLEM SOLUTIONS

(2.1)

$x(t) = Ae^{j(\omega_0 t + \theta)}$, $x(t+T) = e^{j[\omega_0(t+T)+\theta]}$

$= e^{j(\omega_0 t + \theta)} e^{j\omega_0 T}$

For $x(t)$ to be periodic $e^{j\omega_0 T} = 1$ so that $\omega_0 T = 2\pi k$ for k an integer. Thus $T = \frac{2\pi k}{\omega_0}$. Take $k=1$ for fundamental period. Then $T_0 = \frac{2\pi}{\omega_0} \Rightarrow x(t)$ periodic

---

(2.2)

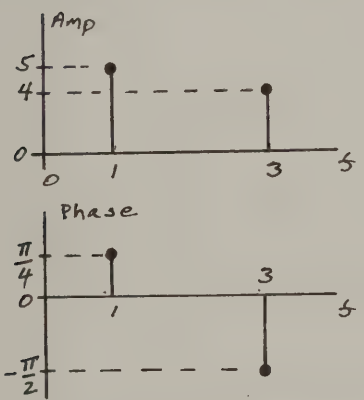
$x(t) = 5\cos(2\pi t + \frac{\pi}{4}) + 4\sin 6\pi t$

$= 5\cos(2\pi t + \frac{\pi}{4}) + 4\cos(6\pi t - \frac{\pi}{2})$

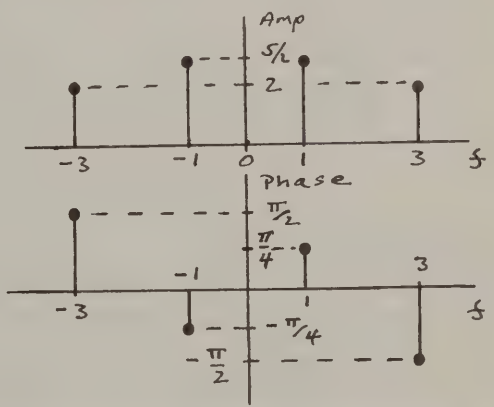
$= \frac{5}{2}\left(e^{j(2\pi t + \frac{\pi}{4})} + e^{-j(2\pi t + \frac{\pi}{4})}\right)$

$+ 2\left(e^{j(6\pi t - \frac{\pi}{2})} + e^{-j(6\pi t - \frac{\pi}{2})}\right)$

Single-Sided Spectra



Double-Sided Spectra



(2.3) From the graph

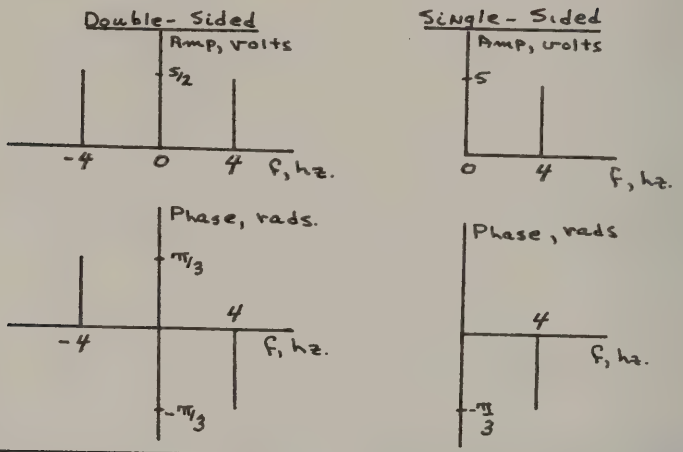
$x(t) = 3e^{j(8\pi t + \frac{\pi}{2})} + 3e^{-j(8\pi t - \frac{\pi}{2})}$

$+ 2e^{j(4\pi t - \frac{\pi}{4})} + 2e^{-j(4\pi t - \frac{\pi}{4})}$

$= 6\cos(8\pi t + \frac{\pi}{2}) + 4\cos(4\pi t - \frac{\pi}{4})$

2.4

$$x(t) = 5\cos(8\pi t - \pi/3) = \mathrm{Re}\left\{5e^{j(8\pi t - \pi/3)}\right\}$$
$$= \frac{5}{2}e^{j(8\pi t - \pi/3)} + \frac{5}{2}e^{-j(8\pi t - \pi/3)}.$$



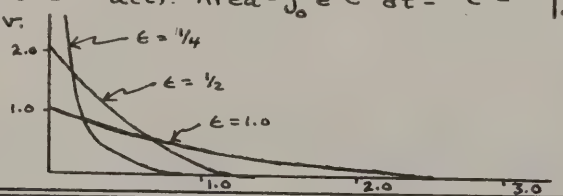
2.5

(a) $\delta_\epsilon(t) = \epsilon\left[\frac{1}{\pi t}\sin\left(\frac{\pi t}{\epsilon}\right)\right]^2$.

$$\text{Area} = \int_{-\infty}^{\infty} \frac{\epsilon}{\pi^2 t^2}\sin^2\left(\frac{\pi t}{\epsilon}\right)dt = \underbrace{\frac{2\epsilon}{\pi^2}\int_0^{\infty} t^{-2}\sin^2\left(\frac{\pi t}{\epsilon}\right)dt}_{\frac{\pi^2}{2\epsilon}\ \text{(see integral tables)}}$$

$\therefore$ Area = 1 for all ϵ.

(b) $\delta_\epsilon(t) = \frac{1}{\epsilon}e^{-t/\epsilon}u(t)$. Area $= \int_0^{\infty}\frac{1}{\epsilon}e^{-t/\epsilon}dt = \frac{-\epsilon}{\epsilon}e^{-t/\epsilon}\Big|_0^{\infty} = 1$.



(2.6) $\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}\left(1 - \left|\frac{t}{\varepsilon}\right|\right), & |t| \le \varepsilon \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = 2\int_0^{\varepsilon} \frac{1}{\varepsilon}\left(1 - \frac{t}{\varepsilon}\right)dt$$

$$= \frac{2}{\varepsilon}\left[\varepsilon - \frac{1}{\varepsilon}\,\frac{1}{2}\,\varepsilon^2\right] = 1$$

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus as $\varepsilon \to 0$ $\delta_\varepsilon(t)$ is a signal of value ∞ at $t = 0$ and 0 otherwise the area of $\delta_\varepsilon(t)$ is one.

(2.7)

(a) $I = \int_{-\infty}^{\infty} (x^2 + 1)\,\delta(2x + 4)\,dx$

Let $\lambda = 2x$

$$I = \int_{-\infty}^{\infty} \left(\tfrac{1}{4}\lambda^2 + 1\right)\delta(\lambda + 4)\,\tfrac{1}{2}\,d\lambda$$

$$= \tfrac{1}{2}\left(\tfrac{1}{4}\lambda^2 + 1\right)\Big|_{\lambda = -4}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{4}(16) + 1\right) = 5/2$$

(b) $I = \int_{-\infty}^{\infty} x^2\left[\delta(2x - 1) + \delta(2x + 1)\right]dx$

once again let $\lambda = 2x$

$$I = \int_{-\infty}^{\infty} \tfrac{1}{4}\lambda^2\left[\delta(\lambda - 1) + \delta(\lambda + 1)\right]\tfrac{1}{2}\,d\lambda$$

$$= \tfrac{1}{2}\,\tfrac{1}{4}\,\lambda^2\Big|_{\lambda = 1} + \tfrac{1}{2}\,\tfrac{1}{4}\,\lambda^2\Big|_{\lambda = -1}$$

$$= \tfrac{1}{2}\,\tfrac{1}{4} + \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{4}$$

2.8

(a) $I = \int_{-\infty}^{\infty} (x^3 + 2x)\,\delta(x-4)\,dx$

$= (x^3 + 2x)\Big|_{x=4} = (4)^3 + 2(4)$

$= 72$

(b) $I = \int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}x\right)\,\delta\left(x+\frac{1}{2}\right)dx$

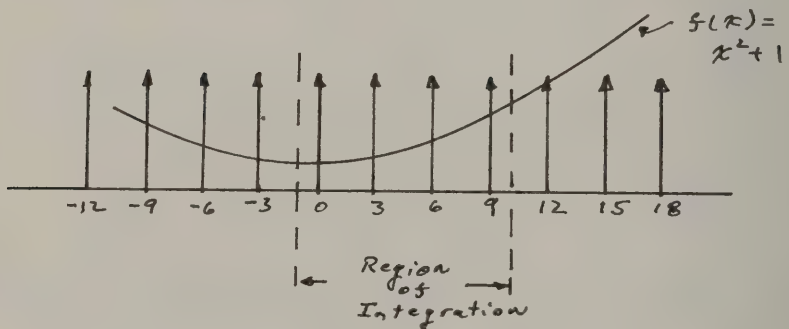
$= \cos\left(\frac{\pi}{2}x\right)\Big|_{x=-\frac{1}{2}}$

$= \cos\left(-\frac{\pi}{4}\right)$

$= 1/\sqrt{2}$

(c)

$$I = \int_{-1}^{10} (x^2+1)\left[\sum_{n=-\infty}^{\infty} \delta(x-3n)\right]dx$$



Thus

$I = (x^2+1)\Big|_{x=0} + (x^2+1)\Big|_{x=3} + (x^2+1)\Big|_{x=6} + (x^2+1)\Big|_{x=9}$

$I = 1 + (9+1) + (36+1) + (81+1)$

$= 1 + 10 + 37 + 82 = 130$

2.9

Given $\int_{-\infty}^{\infty} y(\lambda)\,\delta(\lambda)\,d\lambda = y(0)$

Let $y(t) = x(t+t_0)$

Then $\int_{-\infty}^{\infty} x(\lambda+t_0)\,\delta(\lambda)\,d\lambda = y(0) = x(t_0)$

With the change of variables $u = \lambda + t_0$ we have

$$\int_{-\infty}^{\infty} x(u)\,\delta(u-t_0)\,du = x(t_0)$$

2.10

(a) $x(t) = \underbrace{\cos 5\pi t}_{P_1 = \frac{1}{2.5} = \frac{2}{5}\text{ s.}} + \underbrace{\sin 6\pi t}_{P_2 = \frac{1}{3}\text{ s.}}$ common period = 2 s.

(b) $x(t) = e^{-10t}u(t)$ aperiodic

(c) $x(t) = \underbrace{\sin 2t}_{P_1 = \pi} + \underbrace{\cos \pi t}_{P_2 = 2}$

Since the ratio of P_1 and P_2 cannot be expressed as a ratio of integers, $x(t)$ is <u>not</u> periodic even though it is the sum of two periodic signals.

(d) $x(t) = \Pi\left(\frac{t+3}{7}\right)$ is a single pulse centered at $t = -3$ with width 7. Thus this is aperiodic.

(e) $x(t) = \sum_{n=-\infty}^{\infty} \Pi(t-5n)$ is a periodic pulse train of pulses with width one. The period is 5.

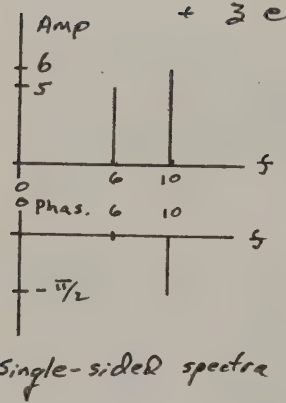
(2.10) (Cont)

(f) $x(t) = 4e^{-4|t|} + 5\cos 6\pi t$

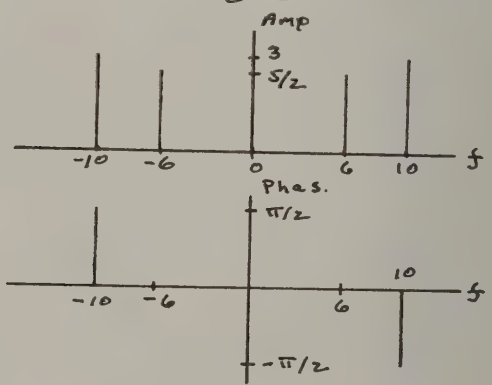
The first term, $4e^{-4|t|}$, is a double-sided exponential and is aperiodic. The second term, $5\cos 6\pi t$, is periodic with period $\frac{1}{3}$. The sum of periodic and aperiodic signals is aperiodic.

(2.11)

$$x(t) = 5\cos 12\pi t + 6\sin 20\pi t$$
$$= 5\cos 12\pi t + 6\cos\left(20\pi t - \frac{\pi}{2}\right)$$
$$= \mathrm{Re}\left\{5e^{j12\pi t} + 6e^{j(20\pi t - \pi/2)}\right\}$$
$$= \frac{5}{2}e^{j12\pi t} + \frac{5}{2}e^{-j12\pi t} + 3e^{j(20\pi t - \pi/2)} + 3e^{j(20\pi t - \frac{\pi}{2})}$$



Single-sided spectra

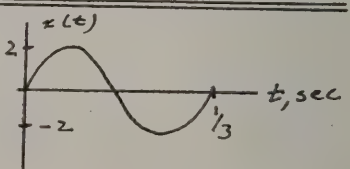


Double-sided Spectra

(2.12)

(a) $x(t) = 2\cos 6\pi t$

$$P_a = \frac{1}{T_0}\int_0^{T_0} 4\cos^2 6\pi t\, dt$$
$$= \frac{1}{T_0}\,4\left(\frac{1}{2}\right)T_0 = 2$$

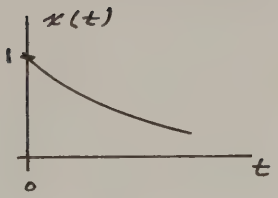


(2.12) (Cont.)

(b) $x(t) = e^{-\alpha t} u(t)$, $\alpha > 0$

$$E = \int_0^\infty (e^{-\alpha t})^2 dt$$

$$= \int_0^\infty e^{-2\alpha t} dt = \frac{1}{2\alpha}$$



(c) $x(t) = e^{\alpha t} u(t)$, $\alpha > 0$

$$E = \lim_{T\to\infty} \int_0^T (e^{\alpha t})^2 dt$$

$$= \lim_{T\to\infty} \int_0^T e^{2\alpha t} dt$$

$$= \lim_{T\to\infty} \frac{1}{2\alpha} (e^{2\alpha T} - 1)$$

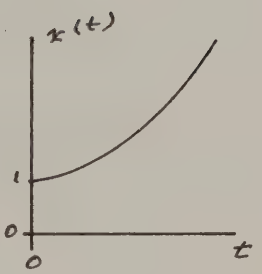
Limit does not exist.

Also

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_0^T e^{2\alpha t} dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} (e^{2\alpha T} - 1)$$

This limit does not exist. Thus $x(t)$ is neither a power nor an energy signal.

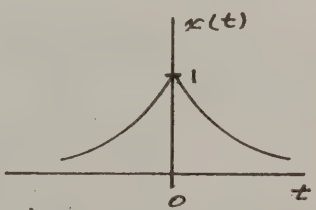


(d) $x(t) = e^{-\alpha|t|}$

$$E = \int_{-\infty}^{\infty} (e^{-\alpha|t|})^2 dt$$

$$= 2\int_0^\infty e^{-2\alpha t} dt$$

$$= \frac{1}{\alpha}$$



2.12 (Cont)

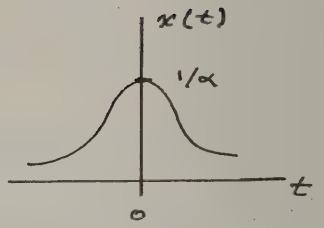
(e) $x(t) = \dfrac{1}{\sqrt{\alpha^2 + t^2}}$

$$E = \int_{-\infty}^{\infty} \frac{dt}{\alpha^2 + t^2}$$

$$= 2\int_{0}^{\infty} \frac{dt}{\alpha^2 + t^2}$$

$$= 2\,\frac{1}{\alpha}\tan^{-1}\frac{t}{\alpha}\Big|_0^{\infty}$$

$$= \frac{2}{\alpha}\,\frac{\pi}{2} = \frac{\pi}{\alpha}$$



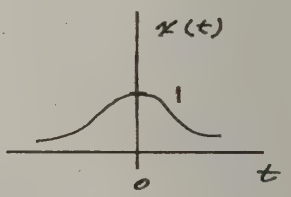
(f) $x(t) = e^{-\alpha^2 t^2}$

$$E = \int_{-\infty}^{\infty} e^{-2\alpha^2 t^2}\,dt$$

$$= 2\int_{0}^{\infty} e^{-2\alpha^2 t^2}\,dt$$

since $\displaystyle\int_0^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$

$$E = 2\,\frac{\sqrt{\pi}}{2\sqrt{2}\alpha} = \frac{1}{\alpha}\sqrt{\frac{\pi}{2}}$$



2.13

(a) $x(t) = A|\cos \omega t|$, $E = \infty$, $P = \frac{1}{2}A^2$

(b) $x(t) = At\,e^{-t/\tau}u(t)$, $\tau > 0$

$$E = \int_0^{\infty} A^2 t^2 e^{-2t/\tau}\,dt \qquad \text{Let } x = \frac{2t}{\tau}$$

$$E = A^2\int_0^{\infty} \frac{\tau^2}{4}\,x^2 e^{-x}\,\frac{\tau}{2}\,dx$$

$$= A^2\,\frac{\tau^3}{8}\,\Gamma(3) = A^2\,\frac{\tau^3}{8}\,2 = \frac{A^2\tau^3}{4}$$

$P = 0$

(2.13) (cont.)

(c) $x(t) = A\tau/(\tau + jt)$

$$|x(t)|^2 = \frac{A^2\tau^2}{\tau^2 + t^2}$$

$$E = A^2\tau^2 \int_{-\infty}^{\infty} \frac{dt}{\tau^2+t^2} = 2A^2\tau^2 \int_0^{\infty} \frac{dt}{\tau^2+t^2}$$

$$= 2A^2\tau^2 \frac{1}{\tau} \tan^{-1}\frac{t}{\tau}\Big|_0^{\infty}$$

$$= 2A^2\tau\left(\frac{\pi}{2}\right) = \pi A^2\tau, \quad P = 0$$

(2.14)

(a) $$P_{x_1} = \frac{1}{T_0}\int_0^{T_0} A^2 \sin^2(\pi t/T_0)\,dt$$

$$= \frac{1}{T_0} A^2 \frac{1}{2}\int_0^{T_0} dt - \frac{1}{T_0} A^2 \frac{1}{2}\int_0^{T_0} \cos\left(\frac{2\pi}{T_0}\right)dt$$

$$= \frac{1}{2}A^2 \text{ Watts}$$

(b) $$P_{x_2} = \frac{1}{T_0}\int_0^{T_0/2} A^2 \sin^2(\pi t/T_0)\,dt$$

$$= \frac{1}{T_0}\frac{1}{2}A^2\int_0^{T_0/2} dt - \frac{1}{T_0}A^2\frac{1}{2}\int_0^{T_0/2}\cos\left(\frac{2\pi}{T_0}\right)dt$$

$$= \frac{A^2}{2T_0}\frac{T_0}{2} - \frac{A^2}{2T_0}\frac{T_0}{2\pi}\sin\frac{2\pi}{T_0}t\Big|_0^{T_0/2}$$

$$= \frac{1}{4}A^2 - \frac{A^2}{4\pi}\left(\sin(\pi) - \sin(0)\right)$$

$$= \frac{1}{4}A^2 \text{ Watts}$$

Note: It should be obvious to the student that P_{x_2} is $\frac{1}{2}P_{x_1}$.

(c) $$P_{x_2} = \frac{1}{T_0}\int_0^{T_0}\left(\frac{A}{T_0}t\right)^2 dt = \frac{A^2}{T_0^3}\frac{1}{3}t^3\Big|_0^{T_0}$$

$$= \frac{A^2}{3T_0^3}T_0^3 = \frac{1}{3}A^2 \text{ Watts}$$

(2.14) (cont.)

(d) clearly $P_{x_4} = \frac{1}{2} P_{x_3} = \frac{1}{6} A^2$ Watts

(e) clearly $P_{x_5} = P_{x_3} = \frac{1}{3} A^2$ Watts

(2.15)

(a) $E = \int_{-\infty}^{\infty} \left[4\pi\left(\frac{t-6}{3}\right)\right]^2 dt$

$= 16 \int_{-\infty}^{\infty} \pi\left(\frac{t-6}{3}\right) dt$

$\pi\left(\frac{t-6}{3}\right) = \begin{cases} 1, & 4.5 < t < 7.5 \\ 0, & \text{otherwise} \end{cases}$

$E = 16 \int_{4.5}^{7.5} dt = 16\left[7.5 - 4.5\right] = 48$ Jou.

$P = 0$

(b) $E = \infty$

$P = \frac{1}{T} \int_0^T (12)^2 \left| e^{j8\pi t} \right|^2 dt$

$= 144 \frac{1}{T_0} \int_0^{T_0} dt = 144$ Watts

(c) $E = \infty$

$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (12)^2 \left| e^{j8\pi t} \right|^2 u^2(t)\, dt$

$= \lim_{T\to\infty} \frac{144}{2T} \int_0^T dt = 72$ Watts

(d) $E = \infty$

$P = \frac{1}{T} \int_0^T (3 + 6\cos 8\pi t)^2 dt$

$= \frac{1}{T} \int_0^T (9 + 36\cos 8\pi t + 36\cos^2 8\pi t) \cdot dt$

(2.15) (Cont.)

$$P = \frac{1}{T}\int_0^T 9\,dt + \frac{1}{T}\int_0^T 36\cos 8\pi t\,dt \quad (\to 0)$$

$$+ \frac{1}{2T}\int_0^T 36\,dt + \frac{1}{2T}\int_0^T 36\cos 16\pi t\,dt \quad (\to 0)$$

$$P = \frac{1}{T}9T + \frac{1}{2T}36T = 9 + 18$$

$$= 27 \text{ watts}$$

(2.16) $$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_1^{T/2} t^{-1/2}\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\,\frac{1}{1/2}\,t^{1/2}\Big|_1^{T/2} = \lim_{T\to\infty}\frac{2}{T}\left[\sqrt{\frac{T}{2}} - 1\right]$$

$$= \lim_{T\to\infty}\left[\sqrt{\frac{2}{T}} - \frac{2}{T}\right] = 0$$

$$E = \lim_{T\to\infty}\int_1^{T/2} t^{-1/2}\,dt = \lim_{T\to\infty}\frac{1}{1/2}\,t^{1/2}\Big|_1^{T/2}$$

$$= \lim_{T\to\infty} 2\left[\sqrt{\frac{T}{2}} - 1\right] \to \infty$$

Thus, $x(t) = t^{-1/4}$, $1 \le t \le \infty$, is neither an energy signal nor a power signal.

(2.17) $x(t)$ is periodic in T_0. Then

$$\int_{(N-1)T_0}^{NT_0} x^2(t)\,dt = \int_{t_0}^{t_0+T_0} x^2(t)\,dt$$

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} x^2(t)\,dt = \lim_{N\to\infty}\frac{1}{NT_0}\left[\int_{-NT_0}^{NT_0} x^2(t)\,dt + \varepsilon\right]$$

where ε represents the end contributions for $|t| > NT_0$. These go to zero as $N \to \infty$

$$P = \lim_{N\to\infty}\frac{1}{2NT_0}\sum_{i=-N}^{N}\int_{(i-1)T_0}^{iT_0} x^2(t)\,dt$$

$$= \lim_{N\to\infty}\frac{1}{2NT_0}\,2N\int_{t_0}^{t_0+T_0} x^2(t)\,dt = \frac{1}{T_0}\int_{t_0}^{t_0+T} x^2(t)\,dt$$

2.18

(a) $P = \frac{1}{T_0}\int_0^{T_0/2} A^2 \sin^2(2\pi t/T_0)\,dt = \frac{A^2}{T_0}\int_0^{T_0/2}\left[\frac{1}{2} - \frac{1}{2}\cos(2\pi t/T_0)\right]dt$

$= \frac{A^2}{2T_0}\left[t - \sin(2\pi t/T_0)\frac{T_0}{2\pi}\right]\Big|_0^{T_0/2} = \frac{1}{4}A^2.$

(b) $P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2 t^2/T_0^2\,dt = \frac{2A^2}{T_0^3}\int_0^{T_0/2} t^2\,dt = \frac{1}{12}A^2.$

(c) $P = \frac{1}{T_0}\int_0^{\tau} A^2\,dt + \frac{1}{T_0}\int_{\tau}^{T_0} B^2\,dt = \frac{1}{T_0}\left[A^2\tau + B^2T_0 - B^2\tau\right]$

$= \frac{1}{T_0}\left[(A^2-B^2)\tau + B^2T_0\right].$

2.19

(a) $\epsilon_N = \int_T |x(t) - \tilde{x}(t)|^2\,dt$

$= \int_T \left[x(t) - \sum_{m=1}^{N} X_m\phi_m(t)\right]\left[x(t) - \sum_{k=1}^{N} X_k\phi_k(t)\right]^* dt$

$= \int_T \left[x(t) - \sum_{m=1}^{N} X_m\phi_m(t)\right]\left[x^*(t) - \sum_{k=1}^{N} X_k^*\phi_k^*(t)\right]dt$

$= \int_T \left[x(t)x^*(t) - \sum_{m=1}^{N} x^*(t)X_m\phi_m(t) - \sum_{k=1}^{N} x(t)X_k^*\phi_k^*(t) + \sum_{m=1}^{N}\sum_{k=1}^{N} X_mX_k^*\phi_m(t)\phi_k^*(t)\right]dt$

$= \int_T x(t)x^*(t)\,dt - \int_T \sum_{m=1}^{N} x^*(t)X_m\phi_m(t)\,dt - \int_T \sum_{m=1}^{N} x(t)X_m^*\phi_m^*(t)\,dt + \int_T \sum_{m=1}^{N}\sum_{k=1}^{N} X_mX_k^*\phi_m(t)\phi_k^*(t)\,dt$

$= \int_T |x(t)|^2\,dt - \sum_{m=1}^{N}\left[X_m^*\int_T x(t)\phi_m^*(t)\,dt + X_m\int_T x^*(t)\phi_m(t)\,dt\right] + \sum_{m=1}^{N}\sum_{k=1}^{N} X_mX_k^*\underbrace{\int_T \phi_m(t)\phi_k^*(t)\,dt}_{C_m\delta_{mk}}$

$\sum_{m=1}^{N}\sum_{k=1}^{N} X_mX_k^*C_m\delta_{mk} = \sum_{m=1}^{N} C_mX_mX_m^* = \sum_{m=1}^{N} C_m|X_m|^2$

$\therefore \int_T |x(t) - \tilde{x}(t)|^2 = \int_T |x(t)|^2\,dt + \sum_{m=1}^{N} C_m|X_m|^2 - \sum_{m=1}^{N}\left[X_m^*\int_T x(t)\phi_m^*(t)\,dt + X_m\int_T x^*(t)\phi_m(t)\,dt\right].$

(b) Equation 2-17 can be written

$\epsilon_N = \int_T |x(t)|^2\,dt + \sum_{m=1}^{N}\left[C_mX_mX_m^* - X_m\int_T x^*(t)\phi_m(t)\,dt - X_m^*\int_T x(t)\phi_m^*(t)\,dt\right]$

$= \int_T |x(t)|^2\,dt + K_1$

(2.19) (Cont.)

Note that

$$\sum_{n=1}^{N} c_n \left| X_n - \frac{1}{c_n}\int_T x(t)\,\phi_n^*(t)\,dt \right|^2$$

$$= \sum_{n=1}^{N} c_n \left[X_n - \frac{1}{c_n}\int_T x(t)\,\phi_n^*(t)\,dt \right]\left[X_n - \frac{1}{c_n}\int_T x(t)\,\phi_n^*(t)\,dt \right]$$

$$= \sum_{n=1}^{N} c_n \left[X_n - \frac{1}{c_n}\int_T x(t)\,\phi_n^*(t)\,dt \right]\left[X_n^* - \frac{1}{c_n}\int_T x^*(t)\,\phi_n(t)\,dt \right]$$

$$= \sum_{n=1}^{N} \left[c_n X_n X_n^* - X_n \int_T x^*(t)\,\phi_n(t)\,dt - X_n^* \int_T x(t)\,\phi_n^*(t)\,dt \right.$$

$$\left. + \frac{1}{c_n}\left\{ \int_T x(t)\,\phi_n^*(t)\,dt \right\}\left\{ \int_T x^*(t)\,\phi_n(t)\,dt \right\} \right]$$

$$= K_1 + \sum_{n=1}^{N} \frac{1}{c_n}\left\{ \int_T x(t)\,\phi_n^*(t)\,dt \right\}\left\{ \int_T x(t)\,\phi_n^*(t)\,dt \right\}^*$$

$$= K_1 + \sum_{n=1}^{N} \frac{1}{c_n}\left| \int_T x(t)\,\phi_n^*(t)\,dt \right|^2$$

Thus

$$K_1 = \sum_{n=1}^{N} c_n \left| X_n - \int_T x(t)\,\phi_n^*(t)\,dt \right|^2 - \sum_{n=1}^{N} \frac{1}{c_n}\left| \int_T x(t)\,\phi_n^*(t)\,dt \right|^2$$

and

$$\epsilon_n = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N} \frac{1}{c_n}\left| \int_T x(t)\,\phi_n^*(t)\,dt \right|^2 + \sum_{n=1}^{N} c_n \left| X_n - \frac{1}{c_n}\int_T x(t)\,\phi_n^*(t)\,dt \right|^2$$

(2.20)

(a) Note that $\phi_1(t) = \begin{cases} 1 & \text{for } 0 < t < T/4 \\ 0 & \text{otherwise} \end{cases}$, $\phi_2(t) = \begin{cases} 1 & \text{for } \frac{T}{4} < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$

$\phi_3(t)$, $\phi_4(t)$ are similar. Also, $C_n = T/4$.

$X_1 = \frac{4}{T}\int_0^{T/4} \frac{1}{T} t\,dt = \frac{2}{T^2} t^2 \Big|_0^{T/4} = \frac{1}{8}$, $X_2 = \frac{2}{T^2} t^2 \Big|_{T/4}^{T/2} = \frac{3}{8}$,

$X_3 = \frac{2}{T^2} t^2 \Big|_{T/2}^{3T/4} = \frac{5}{8}$, $X_4 = \frac{2}{T^2} t^2 \Big|_{3T/4}^{T} = \frac{7}{8}$.

(b) Note that $\phi_1(t) = \begin{cases} 1 & \text{for } 0 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$, $\phi_2(t) = \begin{cases} 1, & \frac{T}{2} < t < T \\ 0, & \text{otherwise} \end{cases}$

Also, $C_n = T/2$.

$X_1 = \frac{2}{T}\int_0^{T/2} \frac{1}{T} t\,dt = \frac{1}{T^2} t^2 \Big|_0^{T/2} = \frac{1}{4}$, $X_2 = \frac{1}{T^2} t^2 \Big|_{T/2}^{T} = \frac{3}{4}$.

(C) For part (a) $ISE = \int_0^T \left(\frac{t}{T}\right)^2 dt - \sum_{i=1}^{4} C_i X_i^2$

$= T/3 - T/4\left[\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right] = 5.208 \times 10^{-3} T.$

For part (b) $ISE = \frac{T}{3} - \frac{T}{2}\left[\frac{1}{16} + \frac{9}{16}\right] = 2.083 \times 10^{-2} T.$

ϵ_N decreases with N.

(2.21)

(a) The basis functions are orthogonal because the functions $\phi_i(t)\phi_j(t)$, $i \neq j$ are odd and limits on integrals are symmetric. Also, functions are orthonormal.

(b) $X_1 = \int_{-1/2}^{1/2} \cos 2\pi t\,dt = \frac{1}{2\pi} \sin 2\pi t \Big|_{-1/2}^{1/2} = 0.$

$X_2 = \frac{1}{2\pi} \sin 2\pi t \Big|_{-1/2}^{0} - \frac{1}{2\pi} \sin 2\pi t \Big|_{0}^{1/2} = 0.$

$X_3 = \frac{-1}{2\pi} \sin 2\pi t \Big|_{-1/2}^{-1/4} - \frac{1}{2\pi} \sin 2\pi t \Big|_{1/4}^{1/2} + \frac{1}{2\pi} \sin 2\pi t \Big|_{-1/4}^{1/4} = \frac{2}{\pi}.$

$ISE = \int_{-1/2}^{1/2} \cos^2 2\pi t\,dt - \sum_{i=1}^{4} C_i X_i^2 = \frac{1}{2} - \frac{1}{\pi^2} = 0.0947.$

(C) $X_1 = \frac{-1}{2\pi} \cos 2\pi t \Big|_{-1/2}^{1/2} = 0$

$X_2 = \frac{-1}{2\pi} \cos 2\pi t \Big|_{-1/2}^{0} + \frac{1}{2\pi} \cos 2\pi t \Big|_{0}^{1/2} = \frac{-2}{\pi}$

$X_3 = \frac{1}{2\pi} \cos 2\pi t \Big|_{-1/2}^{-1/4} - \frac{1}{2\pi} \cos 2\pi t \Big|_{-1/4}^{1/4} + \frac{1}{2\pi} \cos 2\pi t \Big|_{1/4}^{1/2} = 0.$

$ISE = \frac{1}{2} - \frac{4}{\pi^2} = 0.0947.$

(2.22) Let ω_0 in (2.25) be ω_c

(a)
$$\cos \omega_c t = \tfrac{1}{2} e^{j\omega_c t} + \tfrac{1}{2} e^{-j\omega_c t} \qquad X_1 = X_{-1} = \tfrac{1}{2}$$
$$X_n = 0,\ n \neq \pm 1$$

(b)
$$\sin \omega_c t = \tfrac{1}{2j} e^{j\omega_c t} - \tfrac{1}{2j} e^{-j\omega_c t}$$
$$= \tfrac{1}{2} e^{-j\pi/2} e^{j\omega_c t} + \tfrac{1}{2} e^{j\pi/2} e^{-j\omega_c t}$$
$$X_1 = \tfrac{1}{2} e^{-j\pi/2}, \quad X_{-1} = \tfrac{1}{2} e^{j\pi/2}, \quad X_n = 0,\ n \neq \pm 1$$

(c)
$$\cos^2 \omega_c t \sin 5\omega_c t = \tfrac{1}{2}(1 + \cos 2\omega_c t) \sin 5\omega_c t$$
$$= \tfrac{1}{2} \sin 5\omega_c t + \tfrac{1}{2} (\cos 2\omega_c t) \cos(5\omega_c t - \tfrac{\pi}{2})$$
$$= \tfrac{1}{2} \cos(5\omega_c t - \tfrac{\pi}{2}) + \tfrac{1}{4} \cos(7\omega_c t - \tfrac{\pi}{2}) + \tfrac{1}{4} \cos(3\omega_c t - \tfrac{\pi}{2})$$
$$= \tfrac{1}{4} e^{j(5\omega_c t - \frac{\pi}{2})} + \tfrac{1}{4} e^{-j(5\omega_c t - \frac{\pi}{2})} + \tfrac{1}{8} e^{j(7\omega_c t - \frac{\pi}{2})} + \tfrac{1}{8} e^{-j(7\omega_c t - \frac{\pi}{2})} + \tfrac{1}{8} e^{j(3\omega_c t - \frac{\pi}{2})} + \tfrac{1}{8} e^{-j(3\omega_c t - \frac{\pi}{2})}$$
$$X_5 = X_{-5}^* = \tfrac{1}{4} e^{-j\pi/2}$$
$$X_7 = X_{-7}^* = \tfrac{1}{8} e^{-j\pi/2}$$
$$X_3 = X_{-3}^* = \tfrac{1}{8} e^{-j\pi/2}$$
$$X_n \equiv 0, \quad n \neq \pm 3, \pm 5, \pm 7$$

(d)
$$\sin^3 \omega_c t = \cos(\omega_c t - \tfrac{\pi}{2}) \tfrac{1}{2}(1 - \cos 2\omega_c t)$$
$$= \tfrac{1}{2} \cos(\omega_c t - \tfrac{\pi}{2}) - \tfrac{1}{4} \cos(3\omega_c t - \tfrac{\pi}{2}) - \tfrac{1}{4} \cos(\omega_c t + \tfrac{\pi}{2})$$
$$= \tfrac{3}{4} \cos(\omega_c t - \tfrac{\pi}{2}) - \tfrac{1}{4} \cos(3\omega_c t - \tfrac{\pi}{2})$$
$$= \tfrac{3}{8} e^{-j\frac{\pi}{2} + j\omega_c t} + \tfrac{3}{8} e^{j\frac{\pi}{2} - j\omega_c t} - \tfrac{1}{8} e^{-j\frac{\pi}{2} + j3\omega_c t} - \tfrac{1}{8} e^{j\frac{\pi}{2} - j3\omega_c t}$$
$$X_1 = X_{-1}^* = \tfrac{3}{8} e^{-j\pi/2}, \quad X_3 = X_{-3}^* = -\tfrac{1}{8} e^{-j\frac{\pi}{2}}, \quad X_n = 0,\ n \neq 1, 3$$

(2.23)

$x(t) = 2t^2, \ |t| \le 2 \Rightarrow T_0 = 4, \ \omega_0 = \pi/2.$

$$X_n = \frac{1}{4}\int_{-2}^{2} 2t^2 e^{-jn\pi/2 t}\,dt = \frac{1}{4}\int_{-2}^{2} 2t^2\left[\cos n\tfrac{\pi}{2}t - j\sin n\tfrac{\pi}{2}t\right]dt$$

$$= \frac{1}{2}\int_{-2}^{2} t^2 \cos n\tfrac{\pi}{2}t - \frac{j}{2}\int_{-2}^{2} t^2 \sin n\tfrac{\pi}{2}t\,dt$$

The first integrand is even about the origin while the second integrand is odd about the origin. Thus, the expression reduces to

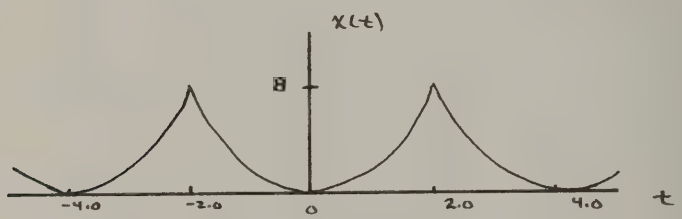
$$X_n = \int_0^2 t^2 \cos\frac{n\pi}{2}t\,dt$$

$$= \left[\frac{2t}{a}\cos at - \frac{2}{a^2}\sin at + \frac{t^2}{a}\sin at\right]_0^2, \quad a = n\pi/2$$

$$= \frac{4}{a^2}\cos 2a = \frac{16}{\pi^2 n^2}\cos n\pi = \frac{16}{n^2\pi^2}(-1)^{|n|}$$

$$X_0 = \frac{1}{4}\int_{-2}^{2} 2t^2 dt = 8/3.$$

$$\therefore \ x(t) = 8/3 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{16}{\pi^2}(-1)^{|n|} e^{jn\pi/2 t} = 8/3 + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi t}{2}\right).$$



(2.24)

(a) $X_N = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jN\omega_0 t}\,dt \Rightarrow X_N^* = \left[\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jN\omega_0 t}\,dt\right]^*$

$= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{jN\omega_0 t}\,dt = X_{-N}.$

$\therefore X_N = X_{-N}^*$. It follows that $|X_N| = |X_{-N}|$ and $\angle X_N = -\angle X_{-N}$.

(b) $X_N = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jN\omega_0 t}\,dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos N\omega_0 t\,dt - j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin N\omega_0 t\,dt$

Since $\sin N\omega_0 t$ is an odd function of t, the second integral is zero. Then since $x(t)$ is real, X_N is real. Since $\cos(N\omega_0 t) = \cos(-N\omega_0 t)$, $X_N = X_{-N}$ and so X_N is even.

(c) Refer to integrals of part (b). Since $x(t)$ is odd and $\cos N\omega_0 t$ is even, the first integral is zero. Then since $x(t)$ is real, X_N is imaginary. Also, since $\sin(-N\omega_0 t) = -\sin(N\omega_0 t)$, X_N is odd.

(d) $X_N = \frac{1}{T_0}\int_0^{T_0} x(t)\cos N\omega_0 t\,dt - j\frac{1}{T_0}\int_0^{T_0} x(t)\sin N\omega_0 t\,dt$.

For N even: $\sin(N\omega_0 t) = \sin\left(N\omega_0 t \pm \frac{N\omega_0 T_0}{2}\right)$

$\cos(N\omega_0 t) = \cos\left(N\omega_0 t \pm \frac{N\omega_0 T_0}{2}\right)$

For N odd: $\sin(N\omega_0 t) = -\sin\left(N\omega_0 t \pm \frac{N\omega_0 T_0}{2}\right)$

$\cos(N\omega_0 t) = -\cos\left(N\omega_0 t \pm \frac{N\omega_0 T_0}{2}\right)$

Thus for N even

$X_N = \frac{1}{T_0}\int_0^{T_0/2} x(t)\cos N\omega_0 t\,dt - \frac{1}{T_0}\int_0^{T_0/2} x(t)\cos N\omega_0 t\,dt$

$- j\frac{1}{T_0}\int_0^{T_0/2} x(t)\sin N\omega_0 t\,dt + j\frac{1}{T_0}\int_0^{T_0/2} x(t)\sin N\omega_0 t\,dt$

$= 0.$

And for N odd

$X_N = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos N\omega_0 t\,dt - j\frac{2}{T_0}\int_0^{T_0/2} x(t)\sin N\omega_0 t\,dt.$

(2.25)

(a) $X_N = \frac{A}{T_0}\int_{t_0-\tau/2}^{t_0+\tau/2} e^{-jN\omega_0 t}\,dt = \frac{-A}{jN\omega_0 T_0}\left[e^{-jN\omega_0(t_0+\tau/2)} - e^{-jN\omega_0(t_0-\tau/2)}\right]$

$= \frac{A}{jN\omega_0 T_0} e^{-jN\omega_0 t_0}\left[e^{jN\omega_0\tau/2} - e^{-jN\omega_0\tau/2}\right] = \frac{2A}{N\omega_0 T_0} e^{-jN\omega_0 t_0} \sin N\omega_0\tau/2$

$= \frac{A\tau}{T_0}\,\mathrm{sinc}(Nf_0\tau)\, e^{-j2\pi N f_0 t_0}$.

(b) $X_N = \frac{A}{T_0}\int_0^{T_0/2} \sin\omega_0 t\, e^{-jN\omega_0 t}\,dt = \frac{-A\omega_0}{T_0\omega_0^2(1-N^2)} e^{-jN\omega_0 t}\left[jN\sin\omega_0 t + \cos\omega_0 t\right]\Big|_0^{\frac{T_0}{2}}$

$= \frac{A}{\omega_0 T_0(1-N^2)}\left(1+e^{-jN\pi}\right) = \begin{cases} \frac{A}{\pi(1-N^2)} & \text{for } N = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise except } N = \pm 1. \end{cases}$

$X_1 = \frac{A}{T_0}\int_0^{T_0/2} \sin\omega_0 t(\cos\omega_0 t - j\sin\omega_0 t)\,dt = -\frac{jA}{4}$.

Similarly, $X_{-1} = \frac{jA}{4}$.

(c) $X_N = \frac{A}{T_0}\int_0^{T_0} |\sin\omega_0 t|\, e^{-jN\omega_0 t}\,dt = \frac{A}{T_0}\int_0^{T_0/2} \sin\omega_0 t\, e^{-jN\omega_0 t}\,dt - \frac{A}{T_0}\int_{T_0/2}^{T_0} \sin\omega_0 t\, e^{-jN\omega_0 t}\,dt$

$= \frac{-A}{\omega_0 T_0(1-N^2)} e^{-jN\omega_0 t}\left[jN\sin\omega_0 t + \cos\omega_0 t\right]\Big|_0^{T_0/2}$

$+ \frac{A}{\omega_0 T_0(1-N^2)} e^{-jN\omega_0 t}\left[jN\sin\omega_0 t + \cos\omega_0 t\right]\Big|_{T_0/2}^{T_0}$

$= \frac{2A}{\omega_0 T_0(1-N^2)}\left(1+e^{-jN\pi}\right) = \begin{cases} \frac{2A}{\pi(1-N^2)} & N = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise except } \pm 1. \end{cases}$

$X_1 = \frac{A}{T_0}\int_0^{T_0} |\sin\omega_0 t|\, e^{-j\omega_0 t}\,dt = 0$ by symmetry.

Similarly, $X_{-1} = 0$.

(d) $X_n = \frac{1}{T_0}\int_{-\frac{1}{2}T_0}^{0} \left(\frac{2A}{T_0}t + A\right) e^{-j2\pi n f_0 t}$

$+ \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} \left(-\frac{2A}{T_0}t + A\right) e^{-j2\pi n f_0 t}\,dt$

$= \frac{2}{T_0}\int_{-\frac{1}{2}T_0}^{0} \left(\frac{2A}{T_0}t + A\right)\cos(2\pi n f_0 t)\,dt$

$= \frac{2}{T_0}\int_{-\frac{1}{2}T_0}^{0} \frac{2A}{T_0}t\cos(2\pi n f_0 t)\,dt$

$+ \frac{2}{T_0}\int_{-\frac{1}{2}T_0}^{0} A\cos(2\pi f_0 t)\,dt$

(2.25) (cont.)

$$X_n = I_1 + I_2 \quad \text{where}$$

$$I_1 = \frac{4A}{T_0^2} \int_{-\frac{1}{2}T_0}^{0} \cos(2\pi n f_0 t)\, dt = \begin{cases} \frac{2A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$I_2 = \frac{2A}{T_0} \int_{-\frac{1}{2}T_0}^{0} \cos(2\pi f_0 t)\, dt = 0$$

Thus

$$X_n = \begin{cases} 2A/\pi^2 n^2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(2.26)

(a) $A_N = \frac{2A}{T_0} \int_{t_0-\tau/2}^{t_0+\tau/2} \cos N\omega_0 t\, dt = \frac{2A}{\omega_0 T_0 N} \sin N\omega_0 t \Big|_{t_0-\tau/2}^{t_0+\tau/2}$

$= \frac{4A}{\omega_0 T_0 N} \cos N\omega_0 t_0 \sin N\omega_0 \tau/2 = \frac{2A\tau}{T_0} \operatorname{sinc} N f_0 \tau \cos 2\pi N f_0 t_0.$

$B_N = \frac{2A}{T_0} \int_{t_0-\tau/2}^{t_0+\tau/2} \sin N\omega_0 t\, dt = \frac{-2A}{\omega_0 T_0 N} \cos N\omega_0 t \Big|_{t_0-\tau/2}^{t_0+\tau/2}$

$= \frac{4A}{\omega_0 T_0 N} \sin N\omega_0 t_0 \sin N\omega_0 \tau/2 = \frac{2A\tau}{T_0} \operatorname{sinc} N f_0 \tau \sin 2\pi N f_0 t_0.$

(b) $A_N = \frac{2A}{T_0} \int_0^{T_0/2} \sin \omega_0 t \cos N\omega_0 t\, dt = \frac{A}{T_0} \int_0^{T_0/2} [\sin(N+1)\omega_0 t + \sin(1-N)\omega_0 t]\, dt$

$= \frac{A}{\omega_0 T_0} \left[\frac{-1}{N+1} \cos(N+1)\omega_0 t + \frac{1}{N-1} \cos(N-1)\omega_0 t \right] \Big|_0^{T_0/2}$

$= \frac{A}{\omega_0 T_0} \left[\frac{2}{1-N^2} - \frac{2}{1-N^2} \cos(N+1)\pi \right] = \begin{cases} \frac{2A}{\pi(1-N^2)} & N = 0, \pm 2, \ldots \\ 0 \text{ otherwise} & \text{except } \pm 1. \end{cases}$

2.26 (CONT.)

$$A_{\pm 1} = \frac{2A}{T_0}\int_0^{T_0/2} \sin\omega_0 t \cos\omega_0 t = \frac{2A}{T_0}\int_0^{T_0/2} \sin 2\omega_0 t\, dt = 0.$$

$$B_N = \frac{2A}{T_0}\int_0^{T_0/2} \sin\omega_0 t \sin N\omega_0 t\, dt = \frac{A}{\omega_0 T_0}\left[\frac{1}{1-N}\sin(1-N)\omega_0 t - \frac{1}{N+1}\sin(1+N)\omega_0 t\right]_0^{\frac{T_0}{2}}$$

$= 0$ for all N except ± 1.

$$B_1 = \frac{2A}{T_0}\int_0^{T_0/2} \sin^2\omega_0 t\, dt = \frac{2A}{T_0}\cdot\frac{T_0}{4} = \frac{A}{2}. \quad \text{Similarly, } B_{-1} = -\frac{A}{2}.$$

(c) $$A_N = \frac{2A}{T_0}\int_0^{T_0} |\sin\omega_0 t| \cos N\omega_0 t\, dt = \frac{2A}{T_0}\int_0^{T_0/2} \sin\omega_0 t \cos N\omega_0 t\, dt$$

$$- \frac{2A}{T_0}\int_{T_0/2}^{T_0} \sin\omega_0 t \cos N\omega_0 t\, dt$$

$$= \frac{A}{\omega_0 T_0}\left[\frac{-1}{N+1}\cos(N+1)\omega_0 t + \frac{1}{N-1}\cos(N-1)\omega_0 t\right]\Big|_0^{T_0/2} - \frac{A}{\omega_0 T_0}[\text{Same term}]\Big|_{T_0/2}^{T_0}$$

$$= \frac{A}{\omega_0 T_0}\left[\frac{2}{N^2-1}\cos(N+1)\pi - \frac{2}{N^2-1} - \frac{2}{N^2-1} + \frac{2}{N^2-1}\cos(N+1)\pi\right]$$

$$= \begin{cases} \frac{4A}{\pi(1-N^2)} & N = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise except } N = \pm 1. \end{cases}$$

$$A_{\pm 1} = \frac{2A}{T_0}\int_0^{T_0/2} \sin\omega_0 t \cos\omega_0 t\, dt - \frac{2A}{T_0}\int_{T_0/2}^{T_0} \sin\omega_0 t \cos\omega_0 t\, dt = 0.$$

$$B_N = \frac{2A}{T_0}\int_0^{T_0/2} \sin\omega_0 t \sin N\omega_0 t\, dt - \frac{2A}{T_0}\int_{T_0/2}^{T_0} \sin\omega_0 t \sin N\omega_0 t\, dt$$

$= 0$ for $N \neq \pm 1$.

$$B_{\pm 1} = \frac{2A}{T_0}\int_0^{T_0/2} \sin^2\omega_0 t\, dt - \frac{2A}{T_0}\int_{T_0/2}^{T_0} \sin^2\omega_0 t\, dt = 0.$$

2.27

$$x(t) = \sum_{N=-\infty}^{\infty} A\,\pi\left(\frac{t - NT_0}{\tau}\right).$$

$$P_{TOTAL} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)\, dt = \frac{A^2}{T_0}\int_{-\tau/2}^{\tau/2} dt = \frac{A^2\tau}{T_0}.$$

From tables, $X_N = \frac{A\tau}{T_0}\,\text{sinc}(Nf_0\tau)$.

Applying Parseval's Theorem: $P = X_0^2 + 2\sum_{N=1}^{\infty} |X_N|^2$

(a) $P_T = \frac{A^2}{2}$. $|Nf_0| \le \frac{1}{\tau} \rightarrow |N| \le \frac{T_0}{\tau} = 2$

$$\therefore P = \frac{A^2}{4}(1 + .81056 + 0) = .45264\,A^2$$

And so $$\frac{P}{P_T} = \frac{.45264}{.5} = .90528.$$

(b) $P_T = \frac{A^2}{10}$. $|N| \le 10$.

$$\therefore P = \frac{A^2}{100}(1 + 8.02826) = .090283\,A^2 \Rightarrow P/P_T = 0.9028.$$

2.28

(a) $Y_N = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-jN\omega_0 t}\,dt = \frac{1}{T_0}\int_0^{T_0} x(t-t_0)e^{-jN\omega_0 t}\,dt$

Change of variables to $t = t' + t_0$. Then

$Y_N = \frac{1}{T_0}\int_{-t_0}^{-t_0+T_0} x(t')e^{-jN\omega_0(t'+t_0)}\,dt'$

↖ Note that we can integrate over any period.

$= e^{-jN\omega_0 t_0}\underbrace{\frac{1}{T_0}\int_{-t_0}^{-t_0+T_0} x(t')e^{-jN\omega_0 t'}\,dt'}_{X_N}$.

$\therefore\ Y_N = X_N e^{-jN\omega_0 t_0}$.

(b) $y(t) = A|\cos\omega_0 t| = A|\sin(\omega_0 t + \pi/2)| = A|\sin\omega_0(t+\frac{\pi}{2\omega_0})|$.

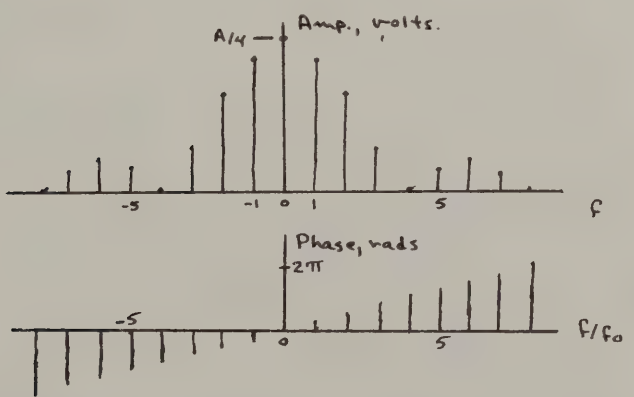
From tables, for $x(t) = A|\sin\omega_0 t|$, $X_N = \begin{cases} \frac{2A}{\pi(1-N^2)}, & N=0,\pm2,\dots \\ 0 \text{ otherwise.} \end{cases}$

$y(t) = x(t+\frac{\pi}{2\omega_0}) \rightarrow t_0 = -\frac{\pi}{2\omega_0}$

$\therefore\ Y_N = \begin{cases} \frac{2A}{\pi(1-N^2)}e^{jN\pi/2} & N=0,\pm2,\dots \\ 0 \quad \text{otherwise.} \end{cases}$

2.29

(a) $t_0 = -\frac{T_0}{8}$, $\tau = \frac{T_0}{4}$. $\therefore\ X_N = \frac{A}{4}\text{sinc}(\frac{N}{4})e^{j\frac{\pi N f_0}{4}}$.



(b) The amplitude spectrum is same as part (a), except $X_0 = \frac{3}{4}A$. Phase has π rad. shift.

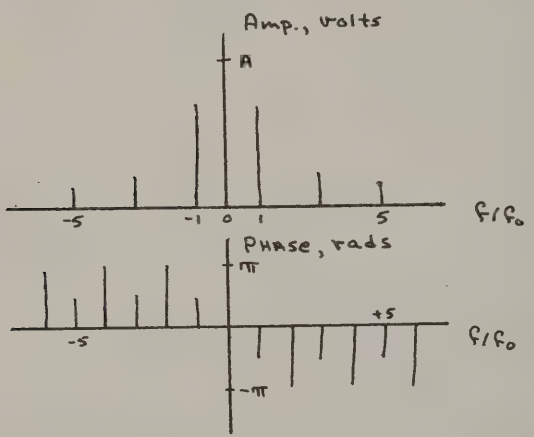
(2.30)

(a) Note that $x_a(t)$ can be written as

$$x_a(t) = 2A \sum_{N=-\infty}^{\infty} \pi\left[\frac{t - T_0/4 - NT_0}{T_0/2}\right] - A$$

$$\therefore X_N = A\,\mathrm{sinc}\left(\frac{N}{2}\right) e^{-j\pi N/2}, \quad N \neq 0$$

$$X_0 = 0.$$



(b) Let X_N^A and X_N^B represent Fourier coefficients of $x_A(t)$ and $x_B(t)$.

$$x_A(t) = K\frac{dx_B(t)}{dt} = K\frac{d}{dt}\sum_{N=-\infty}^{\infty} X_N^B e^{jN\omega_0 t} = K\sum_{N=-\infty}^{\infty} X_N^B \frac{d}{dt} e^{jN\omega_0 t}$$

$$= \sum_{N=-\infty}^{\infty} \underbrace{KjN\omega_0 X_N^B}\, e^{jN\omega_0 t}$$

By uniqueness this must be X_N^A.

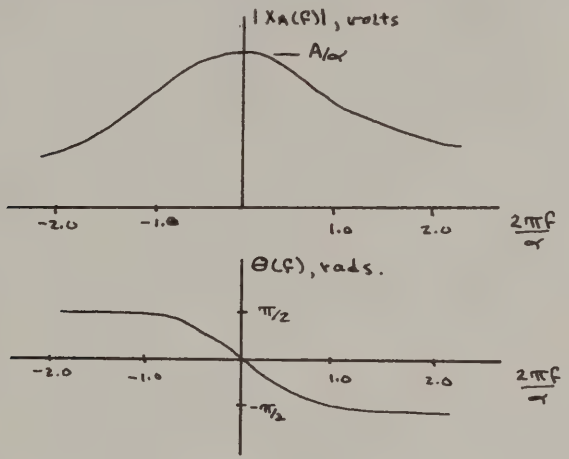
$$\therefore X_N = A\,\mathrm{sinc}\left(\frac{N}{2}\right) e^{-j\pi N/2}\,\frac{1}{jN\omega_0 K}$$

$$= \begin{cases} \dfrac{A}{N\omega_0 K}\,\mathrm{sinc}\left(\dfrac{N}{2}\right) e^{-j(\pi N/2 + \pi/2)}, & N \neq 0 \\ 0 \text{ for } N = 0 \end{cases}$$

(2.31)

(a) $X_A(f) = \int_0^\infty A e^{-\alpha t} e^{-j2\pi ft}\, dt = \frac{A}{\alpha + j2\pi f}$.

$\therefore\ |X_A(f)| = \frac{A}{[\alpha^2 + (2\pi f)^2]^{1/2}}$, $\theta(f) = -\tan^{-1}\left(\frac{2\pi f}{\alpha}\right)$.



(b) $X_B(f) = \int_{-\infty}^0 A e^{\alpha t} e^{-j2\pi ft}\, dt = \frac{A}{\alpha - j2\pi f}$

$\therefore\ |X_A(f)| = \frac{A}{[\alpha^2 + (2\pi f)^2]^{1/2}}$, $\theta(f) = \tan^{-1}\left(\frac{2\pi f}{\alpha}\right)$.

Magnitude plot same as part (a), phase is negative of part (a).

(c) $X_c(f) = \int_{-\infty}^0 A e^{\alpha t} e^{-j2\pi ft}\, dt + \int_0^\infty A e^{-\alpha t} e^{-j2\pi ft}\, dt$

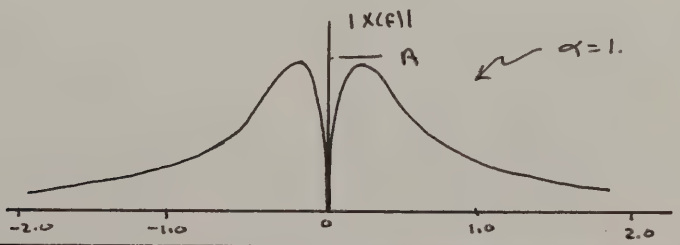
$= \frac{A}{\alpha - j2\pi f} + \frac{A}{\alpha + j2\pi f} = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$.

$\therefore\ |X_c(f)| = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$, $\theta(f) = 0$.

Magnitude plot is squared of part (a), scaled by $\sqrt{\frac{2\alpha}{A}}$.

(d) $X_d(f) = \int_0^\infty A e^{-\alpha t} e^{-j2\pi ft}\, dt - \int_{-\infty}^0 A e^{\alpha t} e^{-j2\pi ft}\, dt = \frac{-2Aj2\pi f}{\alpha^2 + (2\pi f)^2}$.

$\therefore\ |X(f)| = \frac{4A\pi|f|}{\alpha^2 + (2\pi f)^2}$, $\theta(f) = \begin{cases} -\pi/2, & f > 0 \\ +\pi/2, & f < 0 \end{cases}$.



2.32

(a) Let $A=1$, then $\text{sgn}(t) = \lim_{\alpha \to 0} x_d(t)$.

$$\mathcal{F}\{\text{sgn}(t)\} = \mathcal{F}\{\lim_{\alpha\to 0} x_d(t)\} = \lim_{\alpha\to 0} \mathcal{F}\{x_d(t)\} = \lim_{\alpha\to 0} \frac{-4j\pi f}{\alpha^2 + (2\pi f)^2}$$

$$= \frac{-4jf}{(2\pi f)^2} = \frac{-j}{\pi f} = \frac{1}{j\pi f}.$$

(b) $u(t) = \frac{1}{2}\{\text{sgn}(t) + 1\}$.

$$\mathcal{F}\{u(t)\} = \tfrac{1}{2}\mathcal{F}\{\text{sgn}(t)\} + \tfrac{1}{2}\mathcal{F}\{1\} = \frac{1}{j2\pi f} + \tfrac{1}{2}\delta(f).$$

(c) $u(t) = \int_{-\infty}^{t} \delta(t)\,dt \Rightarrow \mathcal{F}\{u(t)\} = \frac{1}{j2\pi f}\mathcal{F}\{\delta(t)\} + \frac{1}{2}\mathcal{F}\{\delta(t)\}\Big|_{f=0}\delta(f)$

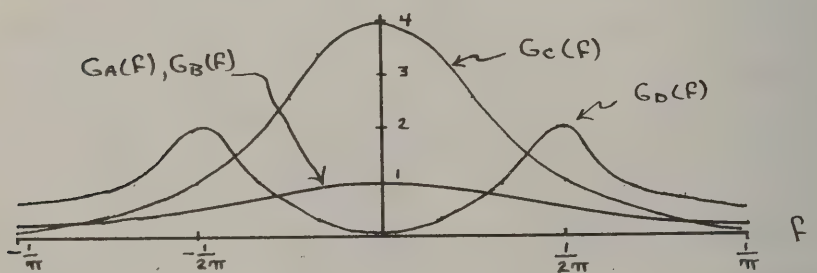
$$= \frac{1}{j2\pi f} + \tfrac{1}{2}\delta(f).$$

2.33

Energy Spectral density $= G(f) = |X(f)|^2$.

$\therefore$ $G_A(f) = G_B(f) = \dfrac{A^2}{\alpha^2 + (2\pi f)^2}$

$G_C(f) = \dfrac{4A^2\alpha^2}{[\alpha^2 + (2\pi f)^2]^2}$, $G_D(f) = \dfrac{16A^2\pi^2 f^2}{[\alpha^2 + (2\pi f)^2]^2}$.



2.34

(a) $\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}\,dt$.

Change of variables, let $t' = t - t_0$. Then

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t')e^{-j2\pi f(t'+t_0)}\,dt = e^{-j2\pi ft_0}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}\,dt}_{X(f)}$$

2.34 (Cont.)

(b) $\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j2\pi f t}\, dt.$

Change of variables, let $t' = at$. Then

$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} \frac{1}{a}\, x(t')\, e^{-j2\pi f t'/a}\, dt' = \frac{1}{a}\, X\left(\frac{f}{a}\right).$

(c) $\int_{-\infty}^{\infty} x(-f)\, e^{j2\pi f}\, df = \underbrace{\int_{-\infty}^{\infty} X(f)\, e^{-j2\pi f t}\, df}_{x(t)}$ (by change of var.)

(d) $\mathcal{F}\{x(t)\cos\omega_0 t\} = \int_{-\infty}^{\infty} x(t)\cos\omega_0 t\, e^{-j2\pi f' t}\, dt$

$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f' t}\, \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] dt$

$= \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f'-f_0)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f'+f_0)t}\, dt$

$= X(f)\big|_{f=f'-f_0} + X(f)\big|_{f=f'+f_0}.$

(e) $y(t) = \int_{-\infty}^{t} x(t')\, dt' = \int_{-\infty}^{\infty} x(t')\, u(t-t')\, dt' = x(t) * u(t)$

$\therefore\ Y(f) = X(f)\,\mathcal{F}\{u(t)\} = X(f)\left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right].$

(f) $x_2(t) = \int_{-\infty}^{\infty} X_2(f')\, e^{j2\pi f' t}\, df'.$

$\mathcal{F}\{x_1(t)\, x_2(t)\} = \mathcal{F}\left\{x_1(t)\int_{-\infty}^{\infty} X_2(f')\, e^{j2\pi f' t}\, df'\right\}$

$= \int_{-\infty}^{\infty} x_1(t)\left[\int_{-\infty}^{\infty} X_2(f')\, e^{j2\pi f' t}\, df'\right] e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} X_2(f)\underbrace{\left[\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi(f-f')t}\, dt\right]}_{X_1(f-f')} df'$

$= X_1(f) * X_2(f).$

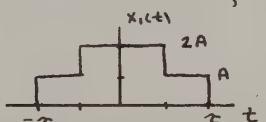
2.35

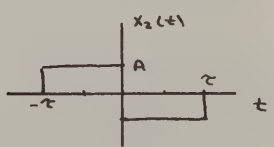
$\mathcal{F}\{x(ct-t_0)\} = \mathcal{F}\left\{x\left[c\left(t-\frac{t_0}{c}\right)\right]\right\} = e^{-j2\pi f t_0/c}\,\mathcal{F}\{x_1(ct)\} \qquad (c>0)$

$= \frac{1}{|c|}\, X(f/c)\, e^{-j2\pi f t_0/c}$

2.36

(a) $x_1(t) = A\,\pi\left(\frac{t}{2\tau}\right) + A\,\pi\left(\frac{t}{\tau}\right)$, $x_2(t) = A\,\pi\left(\frac{t+\tau/2}{\tau}\right) - A\,\pi\left(\frac{t-\tau/2}{\tau}\right)$





(2.36) (Cont.)

(b) Using the results of prob. 2.28

$X_1(f) = 2A\tau \operatorname{sinc} 2\tau f + A\tau \operatorname{sinc} \tau f$

$X_2(f) = A\tau \operatorname{sinc} \tau f \left[e^{j2\pi f\tau/2} - e^{-j2\pi f\tau/2} \right] = j2\pi A\tau^2 f \operatorname{sinc}^2 \tau f .$

(c) $X_1(t) = A \int_{-\infty}^{\infty} [\delta(t+\tau) + \delta(t+\tau/2) - \delta(t-\tau/2) - \delta(t-\tau)]\, dt$

$\therefore X_1(f) = \frac{A}{j2\pi f} \left[e^{j2\pi f\tau} + e^{j\pi f\tau} - e^{-j\pi f\tau} - e^{-j2\pi f\tau} \right]$

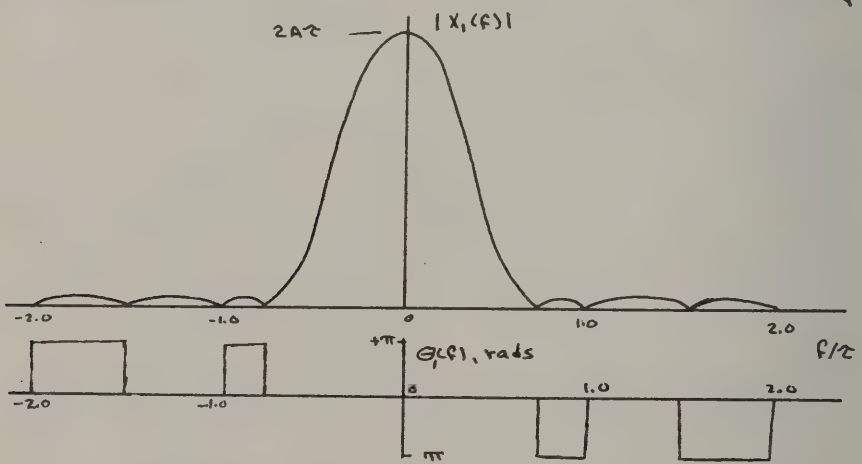
$= \frac{A}{\pi f} [\sin 2\pi f\tau + \sin \pi f\tau] = 2A\tau \operatorname{sinc} 2f\tau + A\tau \operatorname{sinc} f\tau .$

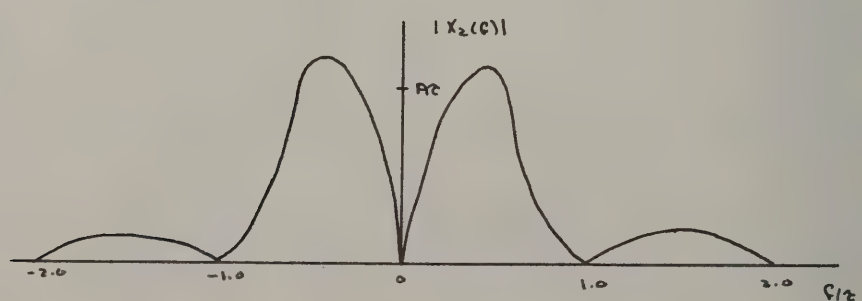
$X_2(t) = A \int_{-\infty}^{\infty} [\delta(t+\tau) - 2\delta(t) + \delta(t-\tau)]\, dt$

$\therefore X_2(f) = \frac{A}{j2\pi f} \left[e^{j2\pi f\tau} + e^{-j2\pi f\tau} - 2 \right] = \frac{j2A}{\pi f} \sin^2 \pi f\tau$

$= j2A\tau \operatorname{sinc} \pi f\tau \sin \pi f\tau .$

(d) $X_1(t)$ is even → $X_1(f)$ is real. $X_2(t)$ is odd → $X_2(f)$ is imag.





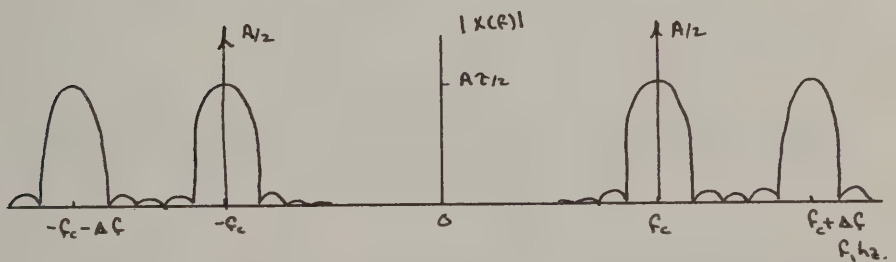
$\Theta_2(f) = \pi/2 \operatorname{sgn}(f).$

2.37

$$x(t) = \underbrace{A\pi(t/\tau)}_{A\tau\,\mathrm{sinc}\,f\tau}\underbrace{\cos 2\pi(f_c+\Delta f)t}_{\frac{1}{2}\delta(f-f_c-\Delta f)+\frac{1}{2}\delta(f+f_c+\Delta f)} + \underbrace{A[1-\pi(t/\tau)]}_{A\delta(f)-A\tau\,\mathrm{sinc}\,f\tau}\underbrace{\cos 2\pi f_c t}_{\frac{1}{2}\delta(f-f_c)+\frac{1}{2}\delta(f+f_c)}$$

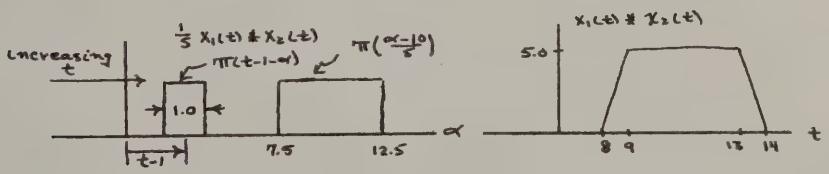
Noting that multiplication in the time domain is equivalent to convolution in the frequency domain and applying sifting property of δ-function gives

$$X(f) = \frac{A\tau}{2}\mathrm{sinc}\,\tau(f-f_c-\Delta f)+\frac{A\tau}{2}\mathrm{sinc}\,\tau(f+f_c+\Delta f) + \frac{A}{2}\delta(f-f_c)+\frac{A}{2}\delta(f+f_c)-\frac{A\tau}{2}\mathrm{sinc}\,\tau(f-f_c)-\frac{A\tau}{2}\mathrm{sinc}\,\tau(f+f_c)$$



2.38

(a) $x_1(t) * x_2(t) = 5\int_{-\infty}^{\infty}\pi(t-1-\alpha)\,\pi\left(\frac{\alpha-10}{5}\right)d\alpha$



(b) $X_1(f) = 5\,\mathrm{sinc}\,f\cdot e^{-j2\pi f}$, $X_2(f) = 5\,\mathrm{sinc}\,5f\cdot e^{-j20\pi f}$

$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(f)X_2(f) = 25\,\mathrm{sinc}\,f\,\mathrm{sinc}\,5f\,e^{-j22\pi f}$

(c) From part (a), $x_1(t) * x_2(t) = 5\int_{-\infty}^{t}[\pi(t-6.5)-\pi(t-11.5)]\,dt$.

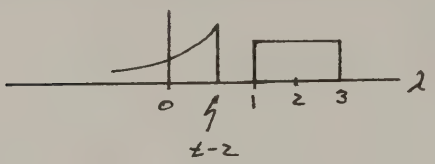
Thus, $j2\pi f\,\mathcal{F}\{x_1(t) * x_2(t)\} = 5[\mathrm{sinc}\,f\,e^{-j17\pi f} - \mathrm{sinc}\,f\,e^{-j27\pi f}]$

So, $\mathcal{F}\{x_1(t) * x_2(t)\} = 25\,\mathrm{sinc}\,f\,\mathrm{sinc}\,5f\,e^{-j22\pi f}$

(2.39)

$x_1(t) = 2e^{-(t-2)}u(t-2)$ $\qquad x_2(t) = \Pi\left(\frac{t-2}{2}\right)$

$$y(t) = \int_{-\infty}^{\infty} x_1(t-\lambda)\, x_2(\lambda)\, d\lambda$$



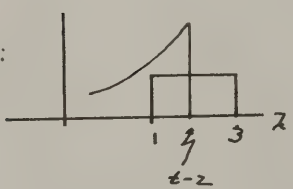
We have three cases of interest; no overlap, partial overlap, and full overlap. The above is case I (no overlap).

$$x_1(t-\lambda) = 2e^{-(t-\lambda-2)}u(t-\lambda-2)$$

Case I: $y(t) \equiv 0$, $\qquad t-2 \le 1$

$t \le 3$

Case II:



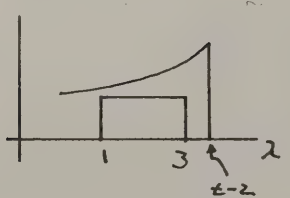
$$y(t) = \int_{1}^{t-2} 2e^{-(t-\lambda-2)}\, d\lambda$$

$$= 2e^{-(t-2)} \int_{1}^{t-2} e^{\lambda}\, d\lambda$$

$$= 2e^{-(t-2)} e^{\lambda}\Big|_{1}^{t-2} = 2e^{-(t-2)}\left[e^{t-2} - e\right]$$

$$= 2\left[1 - e^{-(t-3)}\right], \quad 3 \le t \le 5$$

Case III: $t \ge 5$ $\qquad y(t) = \int_{1}^{3} 2e^{-(t-\lambda-2)}\, d\lambda$



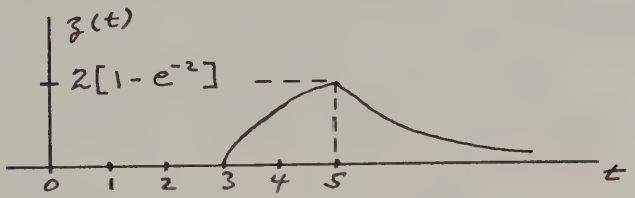
$$= 2e^{-(t-2)} \int_{1}^{3} e^{\lambda}\, d\lambda$$

$$= 2e^{-(t-2)}\left(e^{3} - e\right)$$

$$= 2\left[e^{-(t-5)} - e^{-(t-3)}\right]$$

(2.39) (Cont.)

$$z(t) = 2e^{-t}\left[e^{5} - e^{3}\right], \quad t \geq 5$$



For $Z(f) = X_1(f)\,X_2(f)$

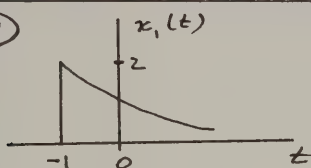
$$X_1(f) = \frac{2}{1 + j2\pi f}\, e^{-j2\pi f(2)}$$

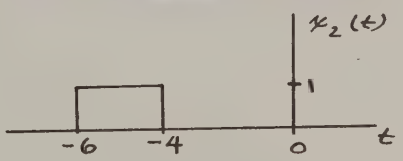
$$X_2(f) = 2\,\mathrm{sinc}\,2f\, e^{-j2\pi f(2)}$$

Thus

$$Z(f) = \frac{4\,\mathrm{sinc}\,2f}{1 + j2\pi f}\, e^{-j8\pi f}$$

(2.40)



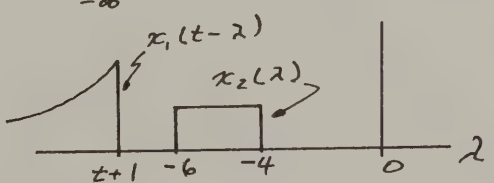


$$x_1(t) = 2e^{-(t+1)}\,u(t+1) \qquad x_2(t) = \Pi\left(\frac{t+5}{2}\right)$$

Again we have three cases; no overlap, partial overlap, and complete overlap.

$$z(t) = \int_{-\infty}^{\infty} x_1(t-\lambda)\,x_2(\lambda)\,d\lambda$$

Case I:

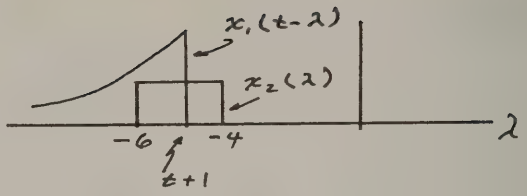


$z(t) = 0, \qquad t + 1 \leq -6$

$t \leq -7$

(2.40) (cont.)

Case II:



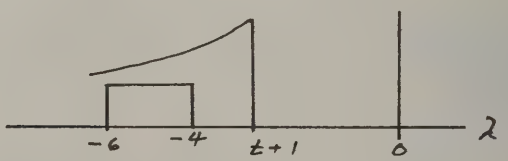
$$z(t) = \int_{-6}^{t+1} 2\, e^{-(t-\lambda+1)}\, d\lambda$$

$$= 2e^{-(t+1)} \int_{-6}^{t+1} e^{\lambda}\, d\lambda$$

$$= 2e^{-(t+1)} \left[e^{(t+1)} - e^{-6} \right]$$

$$= 2\left[1 - e^{-(t+7)} \right] \quad -7 \le t \le -5$$

Case III:



$$z(t) = \int_{-6}^{-4} 2\, e^{-(t-\lambda+1)}\, d\lambda$$

$$= 2e^{-(t+1)} \int_{-6}^{-4} e^{\lambda}\, d\lambda$$

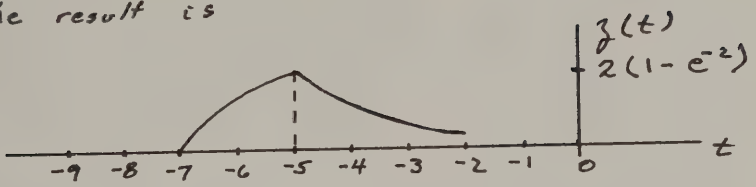
$$= 2e^{-(t+1)} \left[e^{-4} - e^{-6} \right]$$

$$= 2\left[e^{-(t+5)} - e^{-(t+7)} \right], \quad t \ge -5$$

$$= 2e^{-t}\left[e^{-5} - e^{-7} \right], \quad t \ge -5$$

(2.40) (Cont.)

The result is



$$Z(f) = X_1(f)\, X_2(f)$$

$$X_1(f) = \mathcal{F}\{2e^{-(t+1)}\} = \mathcal{F}\{2e^{-t}\}\, e^{j2\pi f}$$

$$= \frac{2}{1+j2\pi f}\, e^{j2\pi f}$$

$$X_2(f) = \mathcal{F}\left\{\pi\left(\frac{t+5}{2}\right)\right\} = \mathcal{F}\left\{\pi\left(\frac{t}{2}\right)\right\} e^{j10\pi f}$$

$$= 2\,\mathrm{sinc}\,2f\; e^{j10\pi f}$$

Thus

$$Z(f) = \frac{4\,\mathrm{sinc}\,2f}{1+j2\pi f}\, e^{j12\pi f}$$

Note that this result could be derived from the previous result by shifting the $z(t)$ found in problem 2.39 to the left 10 sec. This is accomplished by multiplying by $e^{-j2\pi f t_0}$ with $t = -10$.

(2.41)

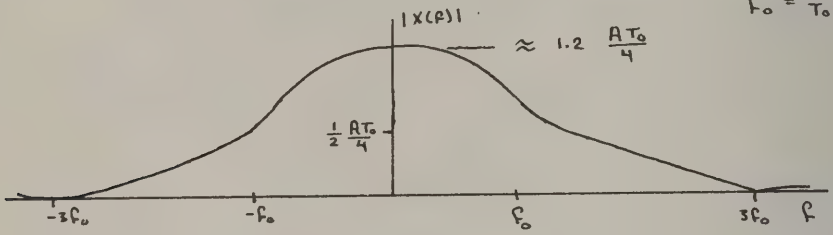
(a) $x(t) = A\,\pi\left(\frac{2t}{T_0}\right)\cos\omega_0 t$.

$\mathcal{F}\left\{A\pi\left(\frac{2t}{T_0}\right)\right\} = \frac{AT_0}{2}\,\text{sinc}\,\frac{T_0 f}{2}$. $\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$.

Applying the multiplication theorem and δ-function sifting property gives

$$X(f) = \frac{AT_0}{4}\left[\text{sinc}\,\tfrac{T_0}{2}(f-f_0) + \text{sinc}\,\tfrac{T_0}{2}(f+f_0)\right] = \frac{AT_0}{4}\left[\text{sinc}\,\tfrac{1}{2}\left(\tfrac{f}{f_0}-1\right) + \text{sinc}\,\tfrac{1}{2}\left(\tfrac{f}{f_0}+1\right)\right]$$

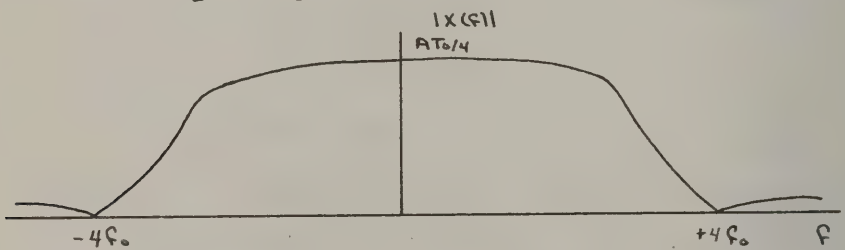
$f_0 = \frac{1}{T_0}$.



(b) $y(t) = \frac{A}{2}\,\pi\left(\frac{2t}{T_0}\right)(1+\cos 2\omega_0 t)$

Superposition and part (a) give

$$Y(f) = \frac{AT_0}{4}\,\text{sinc}\,\tfrac{T_0}{2}f + \frac{AT_0}{8}\left[\text{sinc}\,\tfrac{T_0}{2}(f-2f_0) + \text{sinc}\,\tfrac{T_0}{2}(f+2f_0)\right]$$



(c) ½-Wave Rec. Cosine $= z(t) = \sum_{M=-\infty}^{\infty} x(t - MT_0)$

$\therefore\ \mathcal{F}\{z(t)\} = \sum_{N=-\infty}^{\infty} \frac{1}{T_0}\,X\left(\frac{N}{T_0}\right)\delta\left(f - \frac{N}{T_0}\right)$.

From part (a) $X\left(\frac{N}{T_0}\right) = \frac{AT_0}{4}\left[\text{sinc}\,\frac{T_0}{2}\left(\frac{N}{T_0}-f_0\right) + \text{sinc}\,\frac{T_0}{2}\left(\frac{N}{T_0}+f_0\right)\right]$

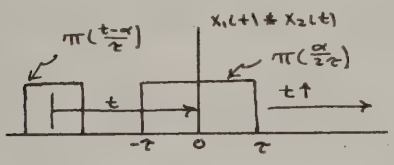
$$= \frac{AT_0}{4}\left[\text{sinc}\left(\tfrac{N-1}{2}\right) + \text{sinc}\left(\tfrac{N+1}{2}\right)\right]$$

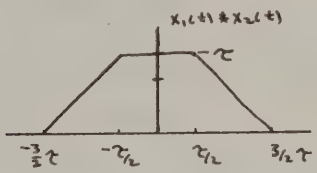
$\therefore\ z(f) = \sum_{N=-\infty}^{\infty} \frac{A}{4}\left[\text{sinc}\left(\tfrac{N-1}{2}\right) + \text{sinc}\left(\tfrac{N+1}{2}\right)\right]\delta\left(f - \frac{N}{T_0}\right)$.

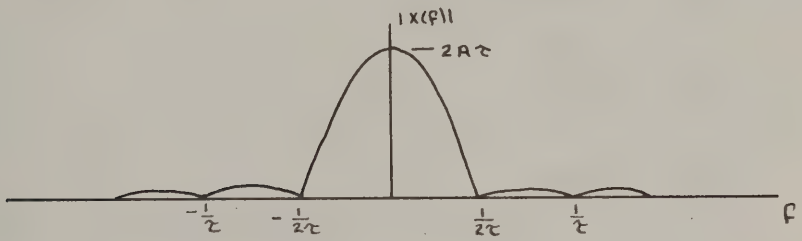
(2.42)

(a) $\mathcal{F}\{\Pi(\frac{t}{\tau})\} = \tau \operatorname{sinc} f\tau$. $\mathcal{F}\{\Pi(\frac{t}{2\tau})\} = 2\tau \operatorname{sinc} 2f\tau$.

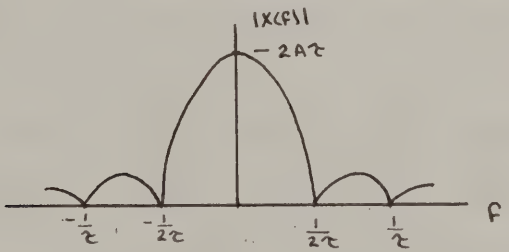
By convolution theorem, $X(f) = 2A\tau \operatorname{sinc} f\tau \operatorname{sinc} 2f\tau$.







(b) $y(t) = A\Pi(\frac{t}{2\tau}) \Rightarrow Y(f) = 2A\tau \operatorname{sinc} 2f\tau$



$y(t)$ has lower frequency sidelobes than $x(t)$ which reflects more gradual increase of $x(t)$ from zero to maximum value.

(2.43)

$$E_A(W) = E_B(W) = \int_{-W}^{W} \frac{A^2}{\alpha^2 + (2\pi f)^2}\,df = \frac{A^2}{2\pi\alpha} \tan^{-1}\left(\frac{2\pi f}{\alpha}\right)\Big|_{-W}^{W}$$

$$= \frac{A^2}{\pi\alpha} \tan^{-1}\frac{2\pi W}{\alpha}. \quad E_{MAX} = \frac{A^2}{2\alpha}.$$

$$E_C(W) = \int_{-W}^{W} \frac{4A^2\alpha^2}{[\alpha^2 + (2\pi f)^2]^2}\,df = \frac{4A^2\alpha^2}{(2\pi)^4} \int_{-W}^{W} \frac{1}{[c^2 + f^2]^2}\,df, \text{ where } c = \frac{\alpha}{2\pi}$$

$$= \frac{A^2 W}{\pi^2\left[\left(\frac{\alpha}{2\pi}\right)^2 + W^2\right]} + \frac{2A^2}{\pi\alpha} \tan^{-1}\left(\frac{2\pi W}{\alpha}\right). \quad E_{MAX} = A^2/\alpha.$$

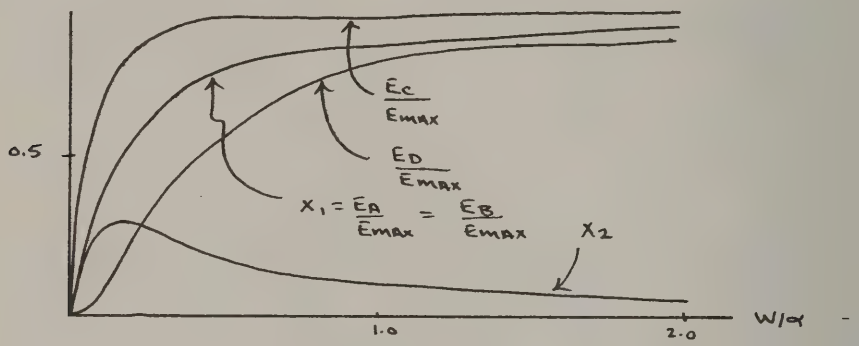
2.43 (cont.)

$$E_D(W) = \int_{-W}^{W} \frac{16A^2\pi^2 f^2}{[\alpha^2+(2\pi f)^2]^2}\,df = \frac{16A^2\pi^2}{(2\pi)^4}\left[\frac{-f}{2(c^2+f^2)} + \frac{1}{2}\int\frac{df}{c^2+f^2}\right]\Bigg|_{-W}^{W}, \quad c = \frac{\alpha}{2\pi}$$

$$= \frac{-A^2 W}{\pi^2\left[\left(\frac{\alpha}{2\pi}\right)^2+W^2\right]} + \frac{2A^2}{\pi\alpha}\tan^{-1}\left(\frac{2\pi W}{\alpha}\right). \quad E_{MAX} = A^2/\alpha.$$

Let $X_1(W) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\pi W}{\alpha}\right)$, $X_2 = \frac{4W/\alpha}{1+\left(\frac{2\pi W}{\alpha}\right)^2}$. Then $\frac{E_A}{E_{MAX}} = \frac{E_B}{E_{MAX}} = X_1$

$\frac{E_C}{E_{MAX}} = X_1 + X_2$, $\frac{E_D}{E_{MAX}} = X_1 - X_2$



2.44

(a) $\int_{-\infty}^{\infty} \mathrm{sinc}^2 a\lambda\,d\lambda = \int_{-\infty}^{\infty} a^2\pi^2(at)\,dt = \int_{-\frac{1}{2a}}^{\frac{1}{2a}} a^2\,dt = a.$

(b) $\int_{-\infty}^{\infty} \mathrm{sinc}^4 a\lambda\,d\lambda = \int_{-\infty}^{\infty} a^2\Lambda^2(at)\,dt = 2a^2\int_{-1/a}^{0}(1+at)^2\,dt$

$= 2a^2\int_{-1/a}^{0}(1+2at+a^2t^2)\,dt = 2a^2\left[t+at^2+\frac{a^2}{3}t^3\right]\Big|_{-1/a}^{0} = \frac{2}{3}a.$

(c) $\int_{-\infty}^{\infty}\frac{1}{(a^2+\lambda^2)^2}\,d\lambda = \int_{-\infty}^{\infty}\frac{(2\pi\tau)^4}{4\tau^2}e^{-2|t|/\tau}$ where $\tau = \frac{1}{2\pi a}$

$= \frac{2\pi^2}{a^2}\int_0^{\infty} e^{-4\pi a t}\,dt = \frac{2\pi^2}{4\pi a^3} = \frac{\pi}{2a^3}.$

2.45

$x(t) * x(-t) = \int_{-\infty}^{\infty} x(-\alpha)\,x(t-\alpha)\,d\alpha$. Change of variables, $\beta = -\alpha$.

Then $x(t) * x(-t) = \int_{-\infty}^{\infty} x(\beta)\,x(t+\beta)\,d\beta$.

$\therefore \lim_{T\to\infty}\frac{1}{2T}\left[x(t) * x(-t)\right]\Big|_{t=\tau} = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$

$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt \triangleq R(\tau).$

(2.46)

Let $y(t) = x(t)\,x(t+\tau)$. Suppose $x(t+T_0) = x(t)$. Then

$y(t+T_0) = x(t+T_0)\,x(t+T_0+\tau) = x(t)\,x(t+\tau) = y(t)$.

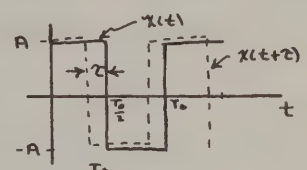
By definition $\langle y(t)\rangle = \frac{1}{T_0}\int_{T_0} y(t)\,dt$.

Since $R(\tau) \triangleq \langle x(t)\,x(t+\tau)\rangle$ it follows that

$R(\tau) = \frac{1}{T_0}\int_{T_0} x(t)\,x(t+\tau)\,dt$ for $x(t)$ periodic.

(2.47)

$R(\tau) = \frac{1}{T_0}\int_0^{T_0} x(t)\,x(t+\tau)\,dt$. Graphically:



$\tau > 0$

$\tau < \frac{T_0}{2}$ $\Longrightarrow$

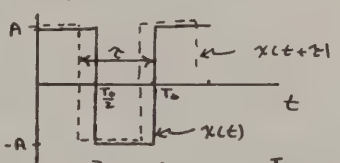
$$\therefore R(\tau) = \frac{1}{T_0}\int_0^{T_0/2-\tau} A^2\,dt + \frac{1}{T_0}\int_{T_0/2-\tau}^{T_0/2} -A^2\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0-\tau} A^2\,dt + \frac{1}{T_0}\int_{T_0-\tau}^{T_0} -A^2\,dt$$

$$= \frac{A^2}{T_0}\left[\frac{T_0}{2}-\tau - \frac{T_0}{2} + \frac{T_0}{2} - \tau + \frac{T_0}{2} - \tau - \frac{T_0}{2} - T_0 + T_0 - \tau\right]$$

$$= A^2\left[1 - 4\,\frac{|\tau|}{T_0}\right] \quad |\tau| < \frac{T_0}{2}$$

Note result is good for $|\tau| < T_0/2$ since $R(-\tau) = R(\tau)$.

Next for $T_0/2 < \tau < T_0$:



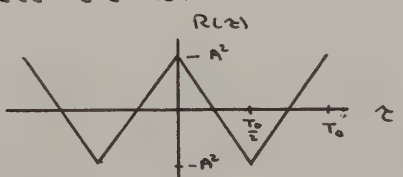
$$\therefore R(\tau) = \frac{1}{T_0}\int_0^{T_0-\tau} -A^2\,dt + \frac{1}{T_0}\int_{T_0-\tau}^{T_0/2} A^2\,dt + \frac{1}{T_0}\int_{T_0/2}^{3/2\,T_0-\tau} -A^2\,dt + \frac{1}{T_0}\int_{3/2\,T_0-\tau}^{T_0} A^2\,dt$$

$$= \frac{A^2}{T_0}\left[-T_0+\tau + T_0/2 - T_0 + \tau - \frac{3}{2}T_0 + \tau + \frac{T_0}{2} + T_0 - \frac{3}{2}T_0 + \tau\right]$$

$$= A^2\left[-3 + 4\,|\tau|/T_0\right] \quad \frac{T_0}{2} < |\tau| < T_0.$$

Thus $$R(\tau) = \begin{cases} A^2\left[1 - 4\,\frac{|\tau|}{T_0}\right] & \text{for } |\tau| < T_0/2 \\ A^2\left[-3 + 4\,\frac{|\tau|}{T_0}\right] & \text{for } T_0/2 < |\tau| < T_0, \end{cases}$$

and is periodic in T_0.



(2.47) (Cont.)

To find $S(f)$, let $P(\tau) = R(\tau)$ for $|\tau| < T_0/2$. Then

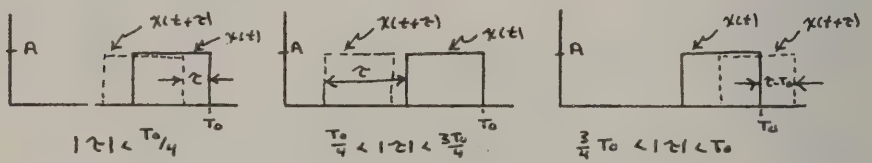
$P(\tau) = 2A^2 \Lambda\left(\frac{2\tau}{T_0}\right) - A^2$ and $P(f) = A^2 T_0 \operatorname{sinc}^2 f\frac{T_0}{2} - A^2\delta(f)$.

Then applying Eqn. 2-55 gives

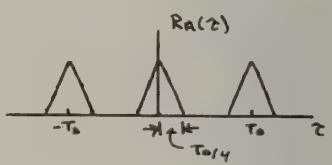
$$S(f) = \sum_{N=-\infty}^{\infty} \frac{1}{T_0}\left[A^2 T_0 \operatorname{sinc}^2\left(\frac{N}{2}\right) - A^2\delta\left(\frac{N}{T_0}\right)\right]\delta\left(f - N\frac{1}{T_0}\right)$$

(2.48)

(a) Graphical solution to $x(t)$ of prob (2.30a):

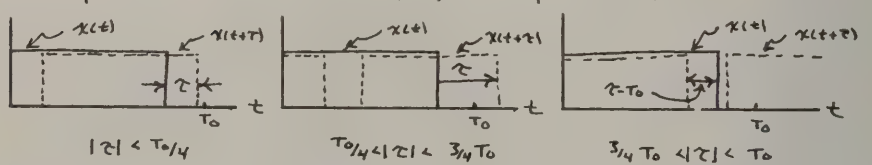


$$\therefore R_A(\tau) = \begin{cases} A^2\left[\frac{1}{4} - \frac{|\tau|}{T_0}\right] & |\tau| < T_0/4 \\ 0 & \frac{T_0}{4} < |\tau| < \frac{3T_0}{4} \\ A^2\left[-\frac{3}{4} + \frac{|\tau|}{T_0}\right] & \frac{3}{4}T_0 < |\tau| < T_0, \end{cases} \Rightarrow$$

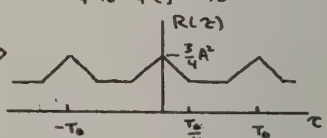


and is periodic in T_0.

(b) Graphical solution to $x(t)$ of prob (2.30b):



$$\therefore R_B(\tau) = \begin{cases} A^2\left[3/4 - \frac{|\tau|}{T_0}\right] & |\tau| < T_0/4 \\ A^2/2 & \frac{T_0}{4} < |\tau| < 3/4\,T_0 \\ A^2\left[-\frac{1}{4} + \frac{|\tau|}{T_0}\right] & 3/4\,T_0 < |\tau| < T_0 \end{cases} \Rightarrow$$



and is periodic in T_0.

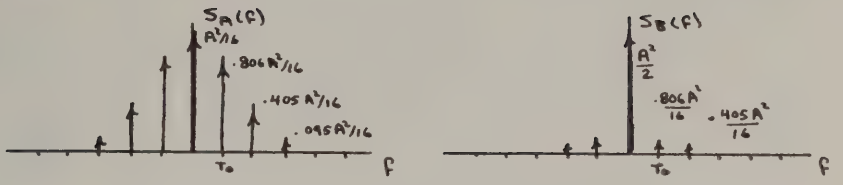
(b) Applying Eqn. 2-55:

$P_A(\tau) = A^2/4\,\Lambda\left(\frac{4\tau}{T_0}\right) \Rightarrow P_A(f) = \frac{A^2 T_0}{16}\operatorname{sinc}^2\frac{T_0 f}{4} \Rightarrow S_A(f) = \sum_{N=-\infty}^{\infty}\frac{A^2}{16}\operatorname{sinc}^2\left(\frac{N}{4}\right)\delta\left(f - \frac{N}{T_0}\right)$

$P_B(\tau) = \frac{A^2}{2} + \frac{A^2}{4}\Lambda\left(\frac{4\tau}{T_0}\right) \Rightarrow P_B(f) = \frac{A^2 T_0}{16}\operatorname{sinc}^2 f\frac{T_0}{4} + A^2/2\,\delta(f),$

$\Rightarrow S_B(f) = \sum_{N=-\infty}^{\infty}\left[\frac{A^2}{16}\operatorname{sinc}^2(N/4)\right]\delta\left(f - \frac{N}{T_0}\right) + A^2/2\,\delta(f).$

(2.48) (Cont.)



(c) Since $x_B(t) = A - x_A(t)$ we can write

$$R_B(\tau) = \frac{1}{T_0}\int_0^{T_0} x_B(t)\,x_B(t+\tau)\,dt = \frac{1}{T_0}\int_0^{T_0}[A - x_A(t)][A - x_A(t+\tau)]\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x_A(t)\,x_A(t+\tau)\,dt + \frac{1}{T_0}\int_0^{T_0} -A\,x_A(t)\,dt + \frac{1}{T_0}\int_0^{T_0} -A\,x_A(t+\tau)\,dt + \frac{1}{T_0}\int_0^{T_0} A^2\,dt$$

$$= R_A(\tau) + A^2 - \frac{2A}{T_0}\int_0^{T_0} x_A(t)\,dt = R_A(\tau) + A^2/2.$$

The level shift results in large DC value in $S_B(f)$. A shift in origin has no effect on correlation.

(2.49)

Assume $y(t) = A(t)e^{-at}$. Then $\frac{dy(t)}{dt} = e^{-at}\left[\frac{dA(t)}{dt} - a\,A(t)\right]$.

Substituting into system equation gives

$$e^{-at}\left[\frac{dA(t)}{dt} - a\,A(t)\right] + a\,A(t)e^{-at} = b\frac{dx(t)}{dt} + c\,x(t).$$

$$\therefore \frac{dA(t)}{dt} = e^{at}\left[b\frac{dx(t)}{dt} + c\,x(t)\right].$$ Integrating gives

$$A(t) = \int_0^t e^{at'}\left[b\frac{dx(t')}{dt'} + c\,x(t')\right]dt' + A(0) = b\int_0^t e^{at'}\frac{dx(t')}{dt'}dt + c\int_0^t e^{at'}x(t')\,dt' + y_0.$$

Using integration by parts:

$$A(t) = b\,e^{at}x(t) - \int_0^t a\,x(t')e^{at'}dt' + \int_0^t c\,x(t')e^{at'}dt' + y_0$$

$$= b\,e^{at}x(t) + (c-ab)\int_0^t x(t')e^{at'}dt' + y_0.$$

$$\therefore y(t) = b\,x(t) + (c-ab)e^{-at}\int_0^t x(t')e^{at'}dt' + y_0\,e^{-at}.$$

By inspection $y_\delta(t) = b\,\delta(t) + (c-ab)e^{-at}$.

Then $y(t) = \int_0^t x(t')\,y_\delta(t-t')\,dt'$.

(2.50) Fourier transforming the d.e. gives

(a)

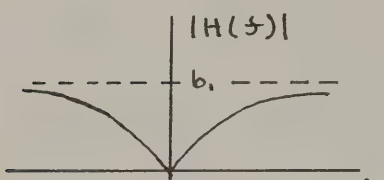
$$Y(f)[j2\pi f + a_0] = X(f)[j2\pi f b_1 + b_0]$$

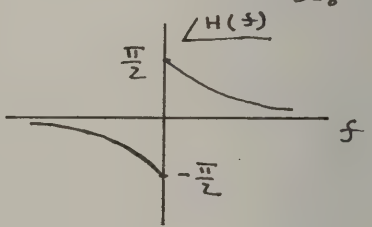
$$H(f) = \frac{b_0 + j2\pi f b_1}{a_0 + j2\pi f} = \frac{Y(f)}{X(f)}$$

(b) With $b_0 = 0$

$$H(f) = \frac{j2\pi f b_1}{a_0 + j2\pi f}$$

$$|H(f)| = \frac{|2\pi b_1 f|}{\sqrt{a_0^2 + (2\pi f)^2}} \qquad \angle H(f) = \frac{\pi}{2}\,\mathrm{sgn}(f) - \tan^{-1}\frac{2\pi f}{a_0}$$

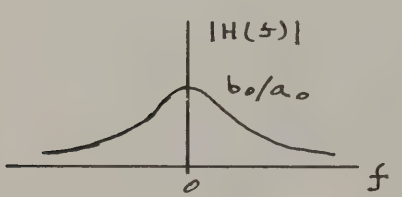


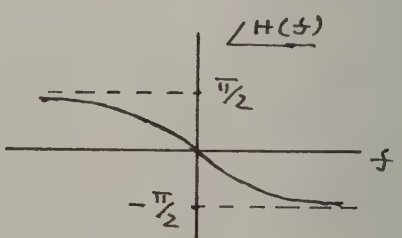


(c) With $b_1 = 0$

$$H(f) = \frac{b_0}{a_0 + j2\pi f}$$

$$|H(f)| = \frac{b_0}{\sqrt{a_0 + (2\pi f)^2}} \qquad \angle H(f) = -\tan^{-1}\frac{2\pi f}{a_0}$$





Note: In this problem it is assumed that $a_0 > 0$ and $b_0 \& b_1 > 0$

(2.51)

(a) $$H_1(f) = \frac{5 + j2\pi f}{10 + j2\pi f}$$

By long division $H_1(f)$ can be expressed

$$H_1(f) = 1 - \frac{5}{10 + j2\pi f}$$

Since $\mathcal{F}\{\delta(t)\} = 1$ and $\mathcal{F}\{Ae^{-\alpha t}u(t)\} = \frac{A}{\alpha + j2\pi f}$

$$h_1(t) = \delta(t) - 5e^{-10t}u(t)$$

(b)

$$H_2(f) = \left[1 + \frac{3}{8 + j6\pi f}\right]e^{-j10\pi f}$$

$$= e^{-j10\pi f} + \frac{1}{\frac{8}{3} + j2\pi f}e^{-j10\pi f}$$

Since $\mathcal{F}\{x(t - t_0)\} = X(f)e^{-j2\pi f t_0}$

$$h_2(t) = \delta(t - 5) + e^{-\frac{8}{3}(t-5)}$$

(2.52)

(a) $y = \frac{R}{R+j2\pi fL} x \rightarrow H(f) = \frac{R}{R+j2\pi fL} = \frac{1}{1+jf/f_3}$, $f_3 = \frac{R}{2\pi L}$.

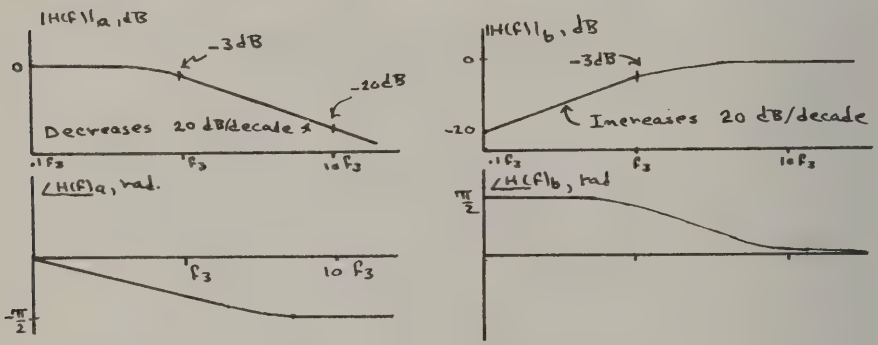
$|H(f)| = [1+(f/f_3)^2]^{-1/2}$, $\angle H(f) = -\tan^{-1}(f/f_3)$; $h(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$

(b) $y = \frac{j2\pi fL}{R+j2\pi fL} x \rightarrow H(f) = \frac{j2\pi fL}{R+j2\pi fL} = \frac{j2\pi f}{R/L + j2\pi f}$.

To find f_3, set $|H(f_3)| = \frac{1}{\sqrt{2}}$:

$\frac{2\pi f_3}{[(R/L)^2 + (2\pi f_3)^2]^{1/2}} = \frac{1}{\sqrt{2}} \Rightarrow (2\pi f_3)^2 = \frac{1}{2}\left(\frac{R}{L}\right)^2 + \frac{1}{2}(2\pi f_3)^2 \Rightarrow f_3 = \frac{R}{2\pi L}$.

$\therefore H(f) = \frac{jf/f_3}{1+jf/f_3}$, $|H(f)| = \frac{|f/f_3|}{[1+(f/f_3)^2]^{1/2}}$, $\angle H(f) = \frac{\pi}{2}\,\mathrm{sgn}(\frac{f}{f_3}) - \tan^{-1}(\frac{f}{f_3})$.



Finding the impulse responses:

For part (a): $\alpha e^{-\alpha t} u(t) \Leftrightarrow \frac{\alpha}{\alpha + j2\pi f}$. So

$h(t) = \mathcal{F}^{-1}\left\{\frac{1}{1+jf/f_3}\right\} = 2\pi f_3 e^{-2\pi f_3 t} u(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$.

For part (b): $(j2\pi f)^n X(f) \Leftrightarrow \frac{d^n}{dt^n} x(t)$. So

$h(t) = \mathcal{F}^{-1}\left\{\frac{jf/f_3}{1+jf/f_3}\right\} = \mathcal{F}^{-1}\left\{\frac{j2\pi fL}{R+j2\pi fL}\right\} = \mathcal{F}^{-1}\left\{j2\pi \frac{L}{R} f\left(\frac{1}{1+j\frac{L}{R}2\pi f}\right)\right\}$.

Using result from part (a):

$\frac{L}{R}\left[\frac{d}{dt}\left(\frac{R}{L} e^{-R/L\,t} u(t)\right)\right] = e^{-R/L\,t}\delta(t) + u(t)\left[-R/L\, e^{-R/L\,t}\right]$

$= h(t) = \delta(t) - \frac{R}{L} e^{-R/L} u(t)$.

(2.53)

(a) $H_1(f) = \dfrac{R_1}{R_1 + j2\pi f L_1} = \dfrac{1}{1 + jf/f_1}$, $f_1 = \dfrac{R_1}{2\pi L_1}$.

$H_2(f) = \dfrac{j2\pi f L_2}{R_2 + j2\pi f L_2} = \dfrac{jf/f_2}{1 + jf/f_2}$, $f_2 = \dfrac{R_2}{2\pi L_2}$.

$H(f) = H_1(f)\, H_2(f) = \dfrac{jf/f_2}{(1 + jf/f_1)(1 + jf/f_2)}$.

(b) Let $s = jf$. Then $H(f) = \dfrac{sf_1}{(s+f_1)(s+f_2)}$.

Set $H(f) = \dfrac{A}{s+f_1} + \dfrac{B}{s+f_2}$. Solving for A, B gives

$A = \dfrac{-f_1^2}{f_2 - f_1}$, $B = \dfrac{-f_1 f_2}{f_1 - f_2}$.

$h(t) = \mathcal{F}^{-1}\left\{\dfrac{A}{f_1 + jf}\right\} + \mathcal{F}^{-1}\left\{\dfrac{B}{f_2 + jf}\right\} = \dfrac{A}{f_1} 2\pi f_1 e^{-2\pi f_1 t}\, u(t)$

$+ \dfrac{B}{f_2} 2\pi f_2 e^{-2\pi f_2 t}\, u(t)$.

$\therefore\ h(t) = \dfrac{R_1}{R_1 L_2 - R_2 L_1}\left[\dfrac{R_1 L_2}{L_1} e^{-\frac{R_1}{L_1}t} - R_2 e^{-\frac{R_2}{L_2}t}\right] u(t)$.

(2.54)

For an ideal integrator $H_I(f) = \dfrac{1}{j2\pi f}$. For a low pass filter $H_{LP}(f) = \dfrac{1}{1 + jf/f_3}$. If we operate far above f_3, then $H_{LP}(f) \approx \dfrac{1}{jf/f_3} \Rightarrow$ Output phase $\approx -90°$, amplitude response inversely proportional to f.

2.55

(a) $$H(f) = \frac{R_2 + j2\pi fL}{R_1 + R_2 + j2\pi fL}$$

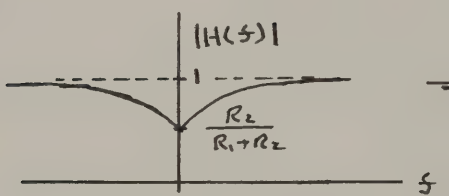
By long division

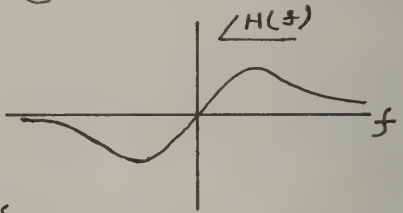
$$H(f) = 1 - \frac{R_1}{R_1 + R_2 + j2\pi fL}$$

$$= 1 - \frac{R_1/L}{\left(\frac{R_1+R_2}{L}\right) + j2\pi f}$$

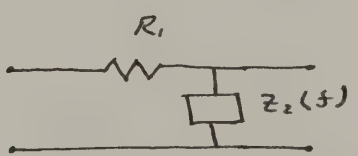
Thus

$$h(t) = \delta(t) - \frac{R_1}{L} e^{-\left(\frac{R_1+R_2}{L}\right)t} u(t)$$





(b)



$$Z_2(f) = \frac{j2\pi fLR_2}{R_2 + j2\pi fL}$$

Thus

$$H(f) = \frac{Z_2(f)}{R + Z_2(f)} = \frac{j2\pi fLR_2/(R_2+j2\pi fL)}{R_1 + j2\pi LR_2/(R_2+j2\pi fL)}$$

$$= \frac{j2\pi fLR_2}{R_1(R_2+j2\pi fL) + j2\pi LR_2}$$

$$= \frac{j2\pi fLR_2}{R_1R_2 + j2\pi f(LR_1 + LR_2)}$$

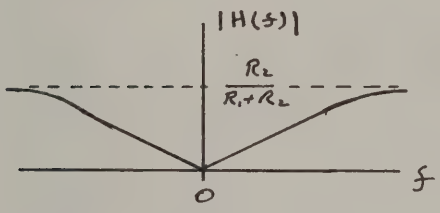
$$= \frac{R_2}{R_1+R_2} \cdot \frac{j2\pi f}{\frac{R_1R_2}{L(R_1+R_2)} + j2\pi f}$$

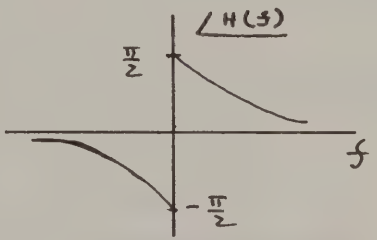
(2.55) (Cont.) By long division.

$$H(f) = \frac{R_2}{R_1+R_2}\left[1 - \frac{R_1R_2}{L(R_1+R_2)}\,\frac{1}{\frac{R_1R_2}{L(R_1+R_2)} + j2\pi f}\right]$$

Thus

$$h(t) = \frac{R_2}{R_1+R_2}\left[\delta(t) - \frac{R_1R_2}{L(R_1+R_2)}\, e^{-\frac{R_1R_2}{L(R_1+R_2)}t}\, u(t)\right]$$





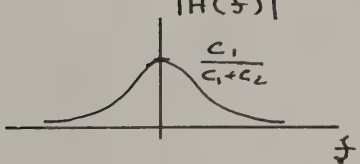
(2.56)

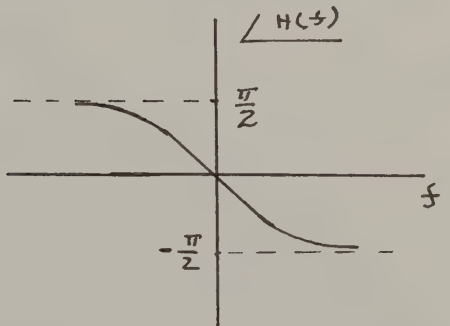
$$H(f) = \frac{\frac{1}{j2\pi f C_2}}{R + \frac{1}{j2\pi f C_1} + \frac{1}{j2\pi f C_2}}$$

$$= \frac{1/C_2}{j2\pi f R + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)}$$

$$= \frac{1}{RC_2}\,\frac{1}{\frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + j2\pi f}$$

$$h(t) = \frac{1}{RC_2}\, e^{-\frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)t}\, u(t)$$





(2.57)

$H(f) = \frac{1}{1+jf/f_3}$, $X(f) = AT\ \text{sinc}\, fT \cdot e^{-j\pi fT}$.

$\therefore\ Y(f) = \frac{AT}{1+jf/f_3}\ \text{sinc}\, fT\, e^{-j\pi fT} = \frac{AT}{1+jf/f_3} e^{-j\pi fT} \frac{\sin \pi fT}{\pi fT}$

$= \frac{AT}{1+jf/f_3} e^{-j\pi fT} \frac{1}{j2\pi fT} \left[e^{j\pi fT} - e^{-j\pi fT}\right]$

$= \frac{A}{j2\pi f(1+jf/f_3)} \left[1 - e^{-j2\pi fT}\right]$.

Using partial fraction expansion:

$\frac{A}{j2\pi f(1+jf/f_3)} = \frac{A}{j2\pi f} - \frac{A}{2\pi f_3}\left[\frac{1}{1+jf/f_3}\right]$

$\therefore\ Y(f) = \frac{A}{j2\pi f} - \frac{A/2\pi f_3}{1+jf/f_3} - \frac{A}{j2\pi f} e^{-j2\pi fT} + \frac{A/2\pi f_3}{1+jf/f_3} e^{-j2\pi fT}$

$y(t) = \frac{A}{2}\text{sgn}\, t - A e^{-j2\pi f_3 t} u(t) - \frac{A}{2}\text{sgn}(t-T) + A e^{-2\pi f_3 (t-T)} u(t-T)$

$\therefore\ y(t) = \begin{cases} A - Ae^{-t/RC} & 0 < t < T \\ Ae^{-(t-T)/RC} - Ae^{-t/RC} & t > T \\ 0 & \text{otherwise} \end{cases}$, $f_3 = \frac{1}{2\pi RC}$.

(2.58)

$X_N = \frac{A}{2}\text{sinc}\frac{N}{2}$. For convenience let $A = 2$, $f_0 = \frac{1}{T_0} = 1$.

Then $x(t) \approx \sum_{N=-4}^{N} \text{sinc}\frac{N}{2} e^{j2\pi Nt}$

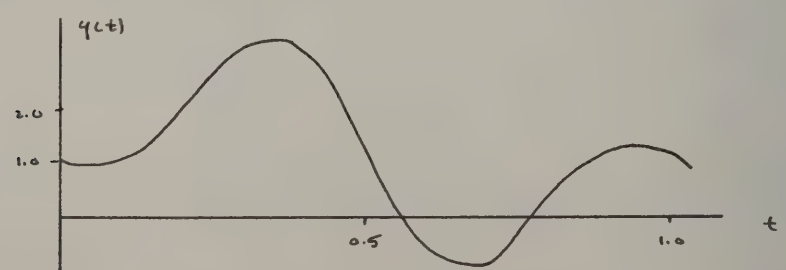
$= 1 + .634\left[e^{j2\pi t} + e^{-j2\pi t}\right] - .212\left[e^{j4\pi t} + e^{-j4\pi t}\right]$

$\therefore\ y(t) = 1 + .634 e^{j2\pi t} e^{-j\pi/2} + .634 e^{-j2\pi t} e^{j\pi/2}$

$- .212 e^{j4\pi t} e^{-j\pi/2} - .212 e^{-j4\pi t} e^{j\pi/2}$

$= 1 + 1.27\cos(2\pi t - \pi/2) - .421\cos(4\pi t - \pi/2)$

$= 1 + 1.27 \sin 2\pi t - .421 \sin 4\pi t$.



(2.58) (Cont.)

Point of problem: There is no change in relative magnitude – only phase. It has a drastic effect on output. Output looks nothing like input.

(2.59)

(a) $y(t) = x(t) + \beta x(t-\tau_m)$. Fourier Transforming gives

$Y(f) = X(f) + \beta X(f) e^{-j2\pi f \tau_m}$.

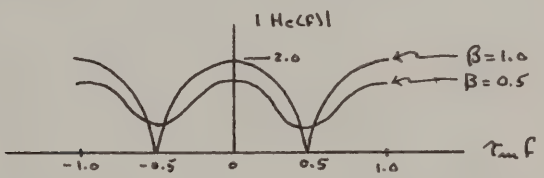
$\therefore H_c(f) = 1 + \beta e^{-j2\pi f \tau_m}$

$|H_c(f)| = [(1+\beta\cos 2\pi f\tau_m)^2 + (\beta \sin 2\pi f \tau_m)^2]^{1/2}$

$= [1 + 2\beta \cos 2\pi f \tau_m + \beta^2]^{1/2}$.

For $\beta = 0.5$: $|H_c(f)| = [1.25 + \cos 2\pi f \tau_m]^{1/2}$

For $\beta = 1.0$: $|H_c(f)| = 2|\cos \pi f \tau_m|$.



(b) $z(t) = \beta_1 y(t) + \sum_{i=2}^{N} \beta_i y[t-(i-1)\Delta]$.

Fourier Transforming gives

$Z(f) = \beta_1 Y(f) + \sum_{i=2}^{N} \beta_i Y(f) e^{-j2\pi f(i-1)\Delta}$

$\therefore H_{eq}(f) = \frac{Z(f)}{Y(f)} = \beta_1 + \sum_{i=2}^{N} \beta_i e^{-j2\pi f(i-1)\Delta}$.

(c) From part (a): $\frac{1}{H_c(f)} = \frac{1}{1+\beta e^{-j2\pi f\tau_m}}$.

So for $\beta < 1$: $\frac{1}{H_c(f)} = 1 - \beta e^{-j2\pi f\tau_m} + \beta^2 e^{-j4\pi f\tau_m} - \beta^3 e^{-j6\pi f\tau_m} + \ldots$

Equating this expression of $\frac{1}{H_c(f)}$ with $H_{eq}(f)$ from part (b) gives

$\beta_1 = 1$, $\beta_i = (-\beta)^{i-1}$.

(2.60)

(a) $y(t) = 2\cos\left(50\pi t - \frac{\pi}{6}\right) + 5\cos\left(120\pi t - \frac{2\pi}{5}\right)$

$= 2\cos 50\pi\left(t - \frac{1}{300}\right) + 5\cos 120\pi\left(t - \frac{1}{300}\right)$

Thus there is amplitude distortion.

(b) $y(t) = \cos\left(120\pi t - \frac{2\pi}{5}\right) + 0.5\cos(160\pi t)$

$= \cos 120\pi\left(t - \frac{1}{300}\right) + 0.5\cos 160\pi(t-0)$

Thus there is phase distortion.

(c) $y(t) = \cos 120\pi\left(t - \frac{1}{300}\right) + 3\cos\left(140\pi t - \frac{7\pi}{15}\right)$

$= \cos 120\pi\left(t - \frac{1}{300}\right) + 3\cos 140\pi\left(t - \frac{1}{300}\right)$

No distortion.

(d) $y(t) = 4\cos\left(20\pi t - \frac{\pi}{15}\right) + 8\cos\left(40\pi t - \frac{2\pi}{15}\right)$

$= 4\cos 20\pi\left(t - \frac{1}{300}\right) + 8\cos 40\pi\left(t - \frac{1}{300}\right)$

No distortion.

(2.61)

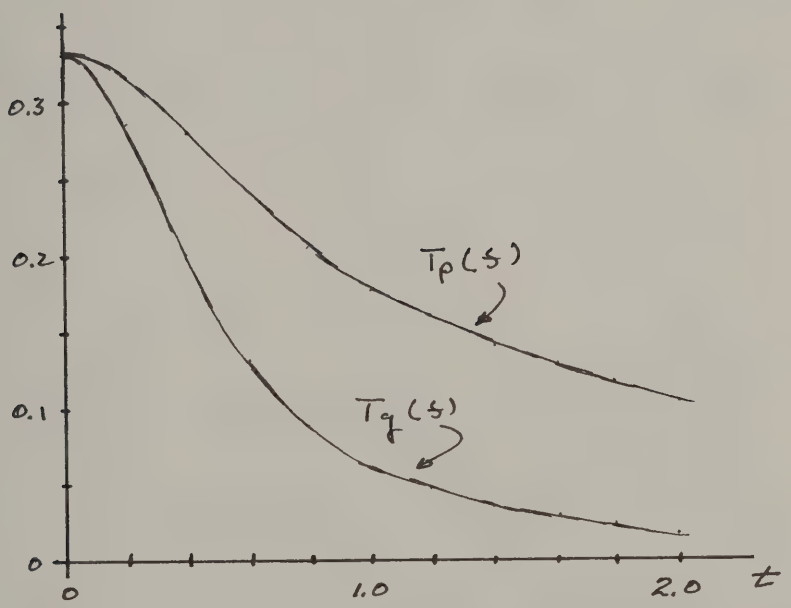
$$h(t) = 5e^{-3t}u(t) \qquad H(f) = \frac{5}{3 + j2\pi f}$$

$$H(f) = \frac{5}{\sqrt{9 + 4\pi^2 f^2}} \angle -\tan^{-1}\frac{2\pi f}{3}$$

$$T_g(f) = -\frac{1}{2\pi}\frac{d}{df}\left(-\tan^{-1}\frac{2\pi f}{3}\right)$$

$$= \frac{1}{2\pi}\frac{1}{1+\left(\frac{2\pi f}{3}\right)^2}\frac{2\pi}{3} = \frac{1}{3}\frac{1}{1+\left(\frac{2\pi f}{3}\right)^2}$$

$$T_p(f) = -\frac{\phi(f)}{2\pi f} = \frac{1}{2\pi f}\tan^{-1}\frac{2\pi f}{3}$$



(2.62)

$$H(f) = \frac{5 + j2\pi f}{10 + j2\pi f}$$

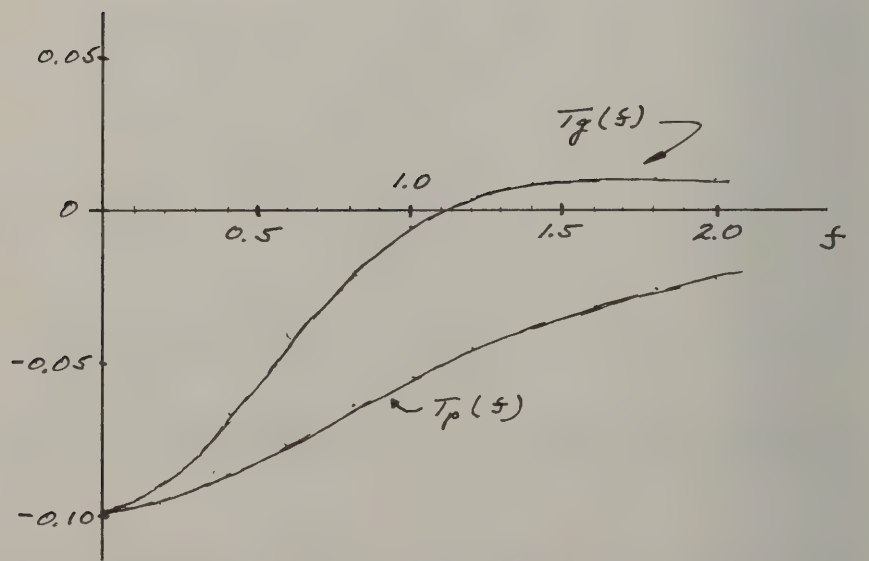
$$= \sqrt{\frac{25 + (2\pi f)^2}{100 + (2\pi f)^2}} \angle \phi(f)$$

where $\phi(f) = \tan^{-1}\frac{2\pi f}{5} - \tan^{-1}\left(\frac{2\pi f}{10}\right)$

$$T_g(f) = -\frac{1}{2\pi}\frac{d\phi}{df} = +\frac{1}{2\pi}\left[\frac{1}{1+\left(\frac{2\pi f}{10}\right)^2}\frac{2\pi}{10} - \frac{1}{1+\left(\frac{2\pi f}{5}\right)^2}\frac{2\pi}{5}\right]$$

$$T_g(f) = \frac{1}{10}\frac{1}{1+\left(\frac{2\pi f}{10}\right)^2} - \frac{1}{5}\frac{1}{1+\left(\frac{2\pi f}{5}\right)^2}$$

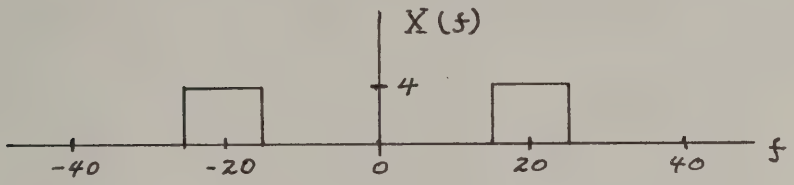
$$T_p(f) = \frac{-\phi(f)}{2\pi f} = \frac{1}{2\pi f}\left[\tan^{-1}\frac{2\pi f}{10} - \tan^{-1}\frac{2\pi f}{5}\right]$$

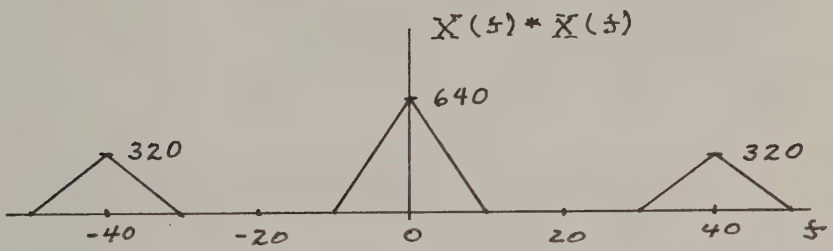


(2.63)

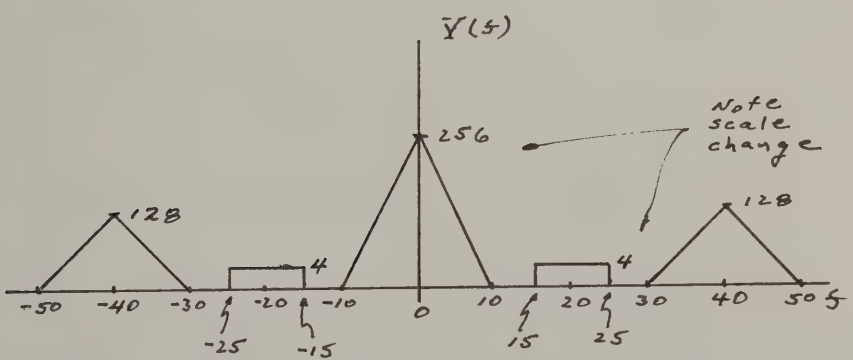
$$y(t) = x(t) + 0.4\,x^2(t)$$

$$Y(f) = X(f) + 0.4\,X(f) * X(f)$$





Now add $X(f)$ and $0.4\,X(f) * X(f)$ to form $Y(f)$.



(2.64)

$$y(t) = \cos 2000\pi t + 0.1\cos^3 2000\pi t$$

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

$$\cos^3 x = \tfrac{1}{2}\cos x + \tfrac{1}{2}\cos 2x \cos x$$

$$= \tfrac{1}{2}\cos x + \tfrac{1}{4}\cos x + \tfrac{1}{4}\cos 3x$$

(2.64) (Cont)

Thus: $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

$$y(t) = \cos 2000\pi t + 0.075\cos 2000\pi t + 0.025\cos 6000\pi t$$

$$= 1.075\cos 2000\pi t + 0.025\cos 6000\pi t$$

Now $H(f) = H_2(f-3000) + H_2(f+3000)$

where $H_2(f) = \dfrac{1}{1+j2Qf}$

Thus for positive frequencies

$$|H(f)| \approx \frac{1}{\sqrt{1+4Q^2(f-3000)^2}}$$

Freq	Amp. of Input	Amp. of Output
1000	1.075	$\dfrac{1.075}{\sqrt{1+4Q^2(4\times10^6)}}$
3000	0.025	0.025

(a) $$THD = \frac{(1.075)^2/[1+16Q^2(10^6)]}{(0.025)^2}$$

$$= (1849)/[1+16Q^2(10^6)]$$

(b) We desire

$$\left\{(1849)/[1+16Q^2(10^6)]\right\} \le 0.001$$

$$1+16Q^2(10^6) \ge 1.849\,(10^6)$$

$$Q^2 \ge \frac{1.849}{16} = 0.116$$

$$Q \ge 0.34$$

(2.65)

$$y(t) = [\cos 10\pi t + \cos 100\pi t] + 0.1[\cos 10\pi t + \cos 100\pi t]^2 + .01[\cos 10\pi t + \cos 100\pi t]^3$$

$$= \cos 10\pi t + \cos 100\pi t + 0.1\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 20\pi t + \tfrac{1}{2} + \tfrac{1}{2}\cos 200\pi t + \cos 110\pi t + \cos 90\pi t\right] + 0.01\left[\cos 10\pi t + \cos 100\pi t + \tfrac{1}{4}\cos 10\pi t + \tfrac{1}{4}\cos 30\pi t + \tfrac{1}{4}\cos 120\pi t + \tfrac{1}{4}\cos 80\pi t + \tfrac{1}{4}\cos 210\pi t + \tfrac{1}{4}\cos 190\pi t + \tfrac{1}{4}\cos 100\pi t + \tfrac{1}{4}\cos 300\pi t + \tfrac{1}{4}\cos 120\pi t + \tfrac{1}{4}\cos 100\pi t + \tfrac{1}{4}\cos 210\pi t + \tfrac{1}{4}\cos 10\pi t + \tfrac{1}{4}\cos 10\pi t + \tfrac{1}{4}\cos 190\pi t + \tfrac{1}{4}\cos 100\pi t + \tfrac{1}{4}\cos 80\pi t\right]$$

(a) At desired freq.: $[\cos 10\pi t + \cos 100\pi t]$

$$+ \underbrace{.0175[\cos 10\pi t + \cos 100\pi t]}_{\text{from intermodulation}}$$

This is not distortion since both terms are scaled the same.

(b) $0.05\,[\cos 20\pi t + \cos 200\pi t]$

(c) $0.0025[\cos 30\pi t + \cos 300\pi t]$

(d) $0.1 + .005\cos 80\pi t + 0.1\cos 90\pi t + 0.1\cos 110\pi t + .005\cos 120\pi t + .005\cos 190\pi t + .005\cos 210\pi t$

(e) $P_{\text{Desired}} = \frac{1}{T}\int_T x^2(t)\,dt = \frac{1}{0.2}\int_0^{0.2}[\cos 10\pi t + \cos 100\pi t]^2\,dt$

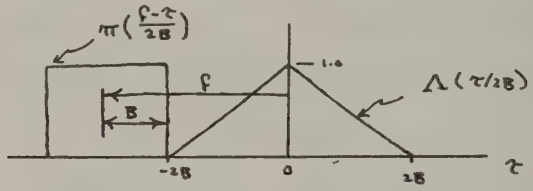
$= \tfrac{1}{2} + \tfrac{1}{2} = 1.0.$

(2.66)

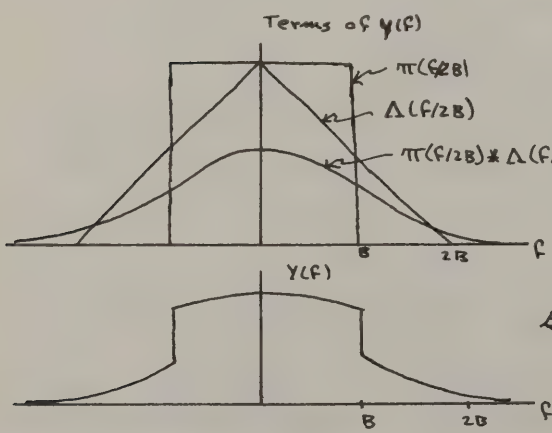
$$Y(f) = A\,\pi(f/2B) + 0.1A^2\,\pi(f/2B) * \pi(f/2B) + 0.01A^3\,\pi(f/2B) * \pi(f/2B) * \pi(f/2B)$$

$$= A\,\pi(f/2B) + 0.2A^2B\,\Lambda(f/2B) + 0.02A^3B\,\Lambda(f/2B) * \pi(f/2B).$$

Graphical convolution of $\Lambda(f/2B) * \pi(f/2B) = Z(f)$:



$$\therefore Z(f) = \begin{cases} 0 & , f < -3B \\ \int_{-2B}^{f+B} \left(1 + \frac{\tau}{2B}\right) d\tau & , -3B < f < -B \\ \int_{f-B}^{f+B} \left(1 - \frac{|\tau|}{2B}\right) d\tau & , -B < f < 0 \\ \text{other values by symmetry} \end{cases} = \begin{cases} 0 & |f| > 3B \\ \frac{1}{4B}\left(f^2 - 6B|f| + 9B^2\right) & -3B < f < -B \\ & B < f < 3B \\ \frac{3}{2}B - \frac{f^2}{2B} & -B < f < 0 \\ & 0 < f < B \end{cases}$$



⇐ Not to Scale.

⇐ Not to Scale.

Contributions of terms to Y(f) depend on A, B.

(2.67)

$$y(t) = a_0 + a_1[\cos\omega_1 t + \cos\omega_2 t] + a_2[1 + \tfrac{1}{2}\cos 2\omega_1 t + \tfrac{1}{2}\cos 2\omega_2 t + \cos(\omega_1-\omega_2)t + \cos(\omega_1+\omega_2)t] + a_3[\cos\omega_1 t + \tfrac{1}{4}\cos\omega_1 t + \tfrac{1}{4}\cos 3\omega_1 t + \tfrac{1}{4}\cos(2\omega_2+\omega_1)t + \tfrac{1}{4}\cos(2\omega_2-\omega_1)t + \tfrac{1}{2}\cos(2\omega_1-\omega_2)t + \tfrac{1}{2}\cos\omega_2 t + \tfrac{1}{2}\cos(2\omega_1+\omega_2)t + \tfrac{1}{2}\cos\omega_2 t + \cos\omega_2 t + \tfrac{1}{4}\cos(\omega_2+2\omega_1)t + \tfrac{1}{4}\cos(2\omega_1-\omega_2)t + \tfrac{1}{4}\cos 3\omega_2 t + \tfrac{1}{4}\cos\omega_2 t + \tfrac{1}{2}\cos(2\omega_2-\omega_1)t + \tfrac{1}{2}\cos\omega_2 t + \tfrac{1}{2}\cos(\omega_1+2\omega_2)t + \tfrac{1}{2}\cos\omega_1 t]$$

∴ Frequencies are present at:

$\omega_1, \omega_2, 0, 2\omega_1, 2\omega_2, \omega_1-\omega_2, \omega_1+\omega_2, 3\omega_1, 2\omega_2+\omega_1,$

$2\omega_2-\omega_1, 2\omega_1-\omega_2, 2\omega_1+\omega_2, 3\omega_2.$

(2.68)

$|H_{BU}(f)| = \dfrac{1}{[1+(f/f_c)^{2N}]^{1/2}}$. Let $R = \left|\dfrac{f}{f_c}\right|$. Then $|H_{BU}(N)| = \dfrac{1}{[1+R^{2N}]^{1/2}}$. Consider $|f| > f_c$. Then $R>1$ and $|H_B(N)|$ is monotonically decreasing function of N, and greatest lower bound is 0.

∴ $\lim_{N\to\infty} |H_{BU}(N)| = 0.$

Similarly, for $|f| < f_c$, $R<1$ and $|H_{BU}(N)|$ is monotonically increasing function of N and least upper bound is 1.

∴ $\lim_{N\to\infty} |H_{BU}(N)| = 1$

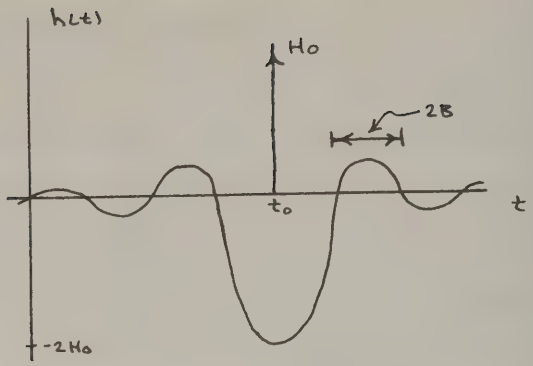
(2.69)

$$H(f) = H_0[1 - \pi(f/2B)]\, e^{-j\omega t_0}$$

$$= H_0 e^{-j\omega t_0} - H_0 \pi(f/2B) e^{-j\omega t_0}$$

∴ $h(t) = H_0\delta(t-t_0) - H_0 2B \operatorname{sinc} 2B(t-t_0).$

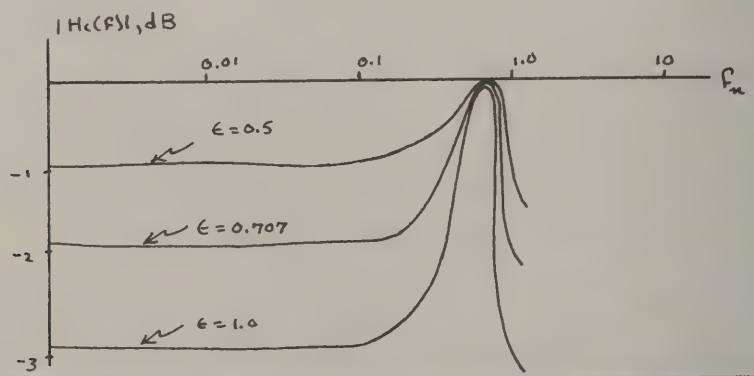
(2.69) (cont.)



(2.70)

$C_0 = 1$, $C_1 = f_n$, $C_2 = 2f_n^2 - 1$, where $f_n = f/f_c$.

$$\therefore |H_c(f)| = \frac{1}{[1 + \epsilon^2(2f_n^2 - 1)^2]^{1/2}}$$



(2.71)

See text for transfer functions.

(2.72)

We define T by (2.116)

$$T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)|\, dt$$

and W by (2.117)

$$W = \frac{1}{2x(0)} \int_{-\infty}^{\infty} |X(f)|\, df$$

and want to verify the inequality

$$2WT \geq 1.$$

(a) $x(t) = Ae^{-t^2/2\tau^2}$

$$X(f) = A\tau e^{-2(\pi f \tau)^2}$$

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} x(t)\, dt = A\tau\sqrt{2\pi}$$

Thus $$T = \frac{A\tau\sqrt{2\pi}}{x(0)} = \tau\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} |X(f)|\, df = \int_{-\infty}^{\infty} X(f)\, df = \frac{A}{\sqrt{2\pi}}$$

Thus

$$W = \frac{A/\sqrt{2\pi}}{2x(0)} = \frac{A/\sqrt{2\pi}}{2A\tau} = \frac{1}{2\sqrt{2\pi}\,\tau}$$

and

$$2WT = 2\left(\frac{1}{2\sqrt{2\pi}\,\tau}\right)(\tau\sqrt{2\pi}) = 1$$

(b) $x(t) = Ae^{-\alpha|t|}$, $X(f) = \dfrac{2A/\alpha}{1 + (2\pi f/\alpha)^2}$

Thus

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} x(t)\, dt = \frac{2A}{\alpha}$$

and so

$$T = \frac{2A/\alpha}{x(0)} = \frac{2}{\alpha}.$$

$$\int_{-\infty}^{\infty} |X(f)|\, df = \int_{-\infty}^{\infty} X(f)\, df = A$$

and so

$$W = \frac{A}{2(2A/\alpha)} = \frac{\alpha}{4}$$

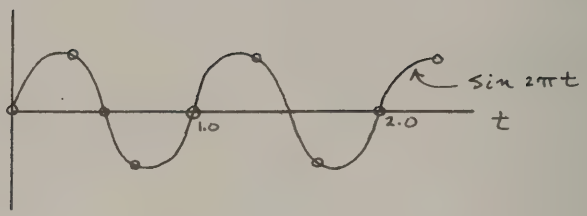
Therefore

$$2WT = 2\left(\frac{\alpha}{4}\right)\left(\frac{2}{\alpha}\right) = 1.$$

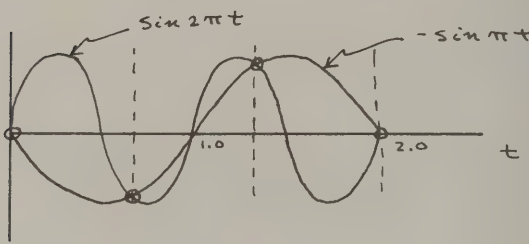
(2.73)

(a) $\frac{1}{2}$ sec

(b)

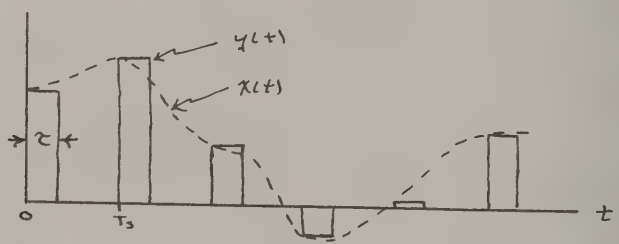


(c)



(2.74)

(a)



(b) $x_s(t) = \sum_{N=-\infty}^{\infty} x(NT_s)\,\delta(t-NT_s)$

$y(t) = x_s(t) * \pi\left(\frac{t-\tau/2}{\tau}\right) = \pi\left(\frac{t-\tau/2}{\tau}\right) * \sum_{N=-\infty}^{\infty} x(NT_s)\,\delta(t-NT_s)$

$= \sum_{N=-\infty}^{\infty} \pi\left[\frac{t-\tau/2-NT_s}{\tau}\right] x(NT_s)$

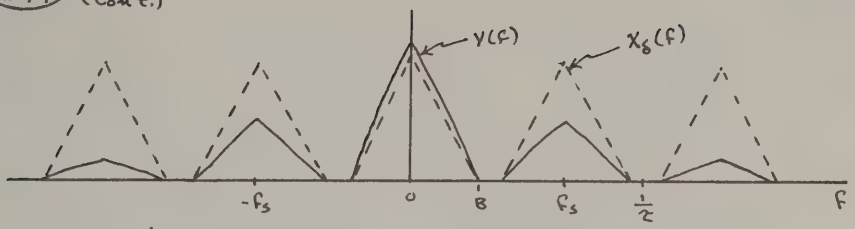
We know that $X_s(f) = f_s \sum_{N=-\infty}^{\infty} X(f-Nf_s)$

Thus $Y(f) = X_s(f)\,H(f)$

$= \tau f_s \operatorname{sinc} \tau f\, e^{-j\omega\tau/2} \sum_{N=-\infty}^{\infty} X(f-Nf_s).$

Let B = signal bandwidth, $T_s = 1/f_s$ = sample space. Then the spectra are

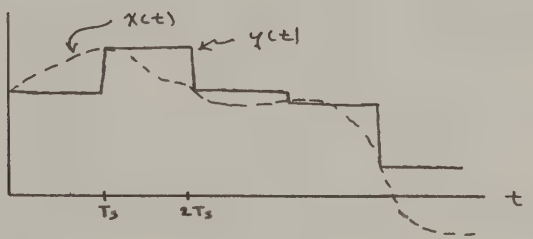
(2.74) (cont.)



Note that we assume $f_s \geq 2B$.
Very little distortion will occur if $\frac{1}{\tau} \gg B$. Note that phase is linear for $|f| < 1/\tau$ so no phase distortion occurs if $\frac{1}{\tau} > B$. Thus $\tau \ll T_s$.

(2.75)

(a)

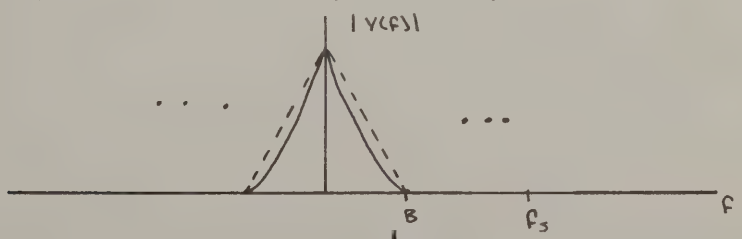


From figure it is apparent that we should choose T_s so that $x(t)$ can change value very little in one sampling interval.

(b) $X_s(f) = \frac{1}{T_s} \sum_{N=-\infty}^{\infty} X(f - \frac{N}{T_s})$, $H(f) = T_s \operatorname{sinc} T_s f \, e^{-j\pi T_s f}$

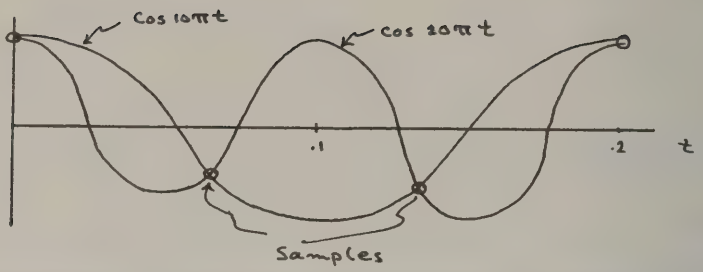
$\therefore \; Y(f) = e^{-j\pi T_s f} \operatorname{sinc} T_s f \sum_{N=-\infty}^{\infty} X(f - \frac{N}{T_s})$.

Assume $X_s(f)$ as in prob. (2.74). Then



For small distortion, $T_s \gg \frac{1}{B}$.

(2.76)

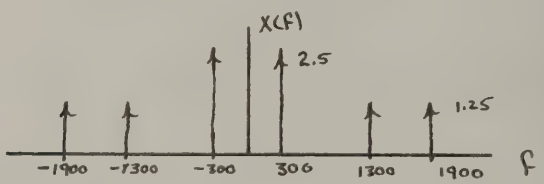


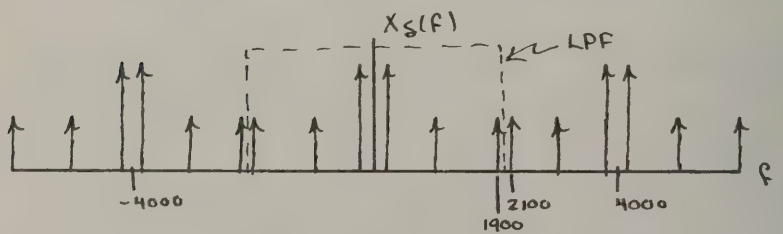
(2.77)

$$x(t) = 10\cos(600\pi t)\cos^2(1600\pi t)$$

$$= 10\cos(600\pi t)\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 3200\pi t\right]$$

$$= 5\cos(600\pi t) + 2.5\cos 3800\pi t + 2.5\cos 2600\pi t.$$

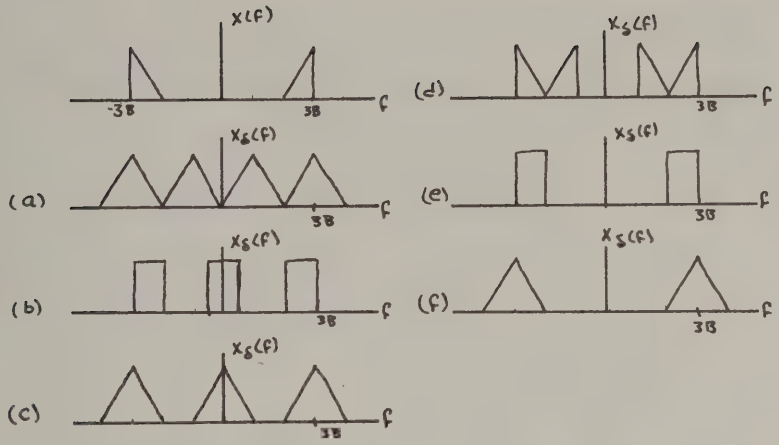




From the figure it is seen that f_{co}, the filter cutoff frequency must satisfy $1900 < f_{co} < 2100$.

The minimum sampling frequency is twice the highest frequency and so is $f_{s,min} = 2 \times 1900 = 3800$ Hz.

(2.78)



For bandpass sampling all but (b) and (e) are satisfactory. For lowpass only (f) will work.

(2.79)

$x(t) = 5\cos(2000\pi t)\cos(8000\pi t) = 2.5\left[\cos(10000\pi t) + \cos(6000\pi t)\right]$

(a) Highest frequency = 5Khz $\Rightarrow$ Sample at 10 Khz.

(b) $B = 2$Khz, $f_u = 5$Khz $\rightarrow M = 2 \Rightarrow$ Sample at 5 Khz.

(2.80)

$$\widehat{\cos\omega_c t} = \int_{-\infty}^{\infty} \frac{1}{\pi(t-t')}\cos\omega_c t'\,dt' = \int_{-\infty}^{\infty} \frac{1}{\pi(t+t')}\cos\omega_c t'\,dt'$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi t'}\cos\omega_c(t'-t)\,dt' = \int_{-\infty}^{\infty} \frac{1}{\pi t'}\left[\cos\omega_c t\cos\omega_c t' + \sin\omega_c t\sin\omega_c t'\right]dt'$$

$$= \cos\omega_c t \underbrace{\int_{-\infty}^{\infty} \frac{1}{\pi t'}\cos\omega_c t'\,dt'}_{\text{Integrand odd so integral is zero.}} + \sin\omega_c t \underbrace{\int_{-\infty}^{\infty} \frac{1}{\pi t'}\sin\omega_c t'\,dt'}_{\text{This integral} = 1 \text{ from tables}}$$

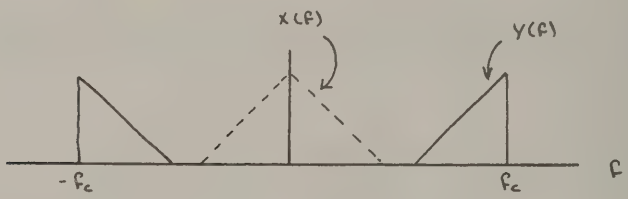
$\therefore \quad \widehat{\cos\omega_c t} = \sin\omega_c t.$

(2.81)

$$y(t) = \underbrace{x(t)\cos\omega_c t}_{\frac{1}{2}X(f-f_c)+\frac{1}{2}X(f+f_c)} + \underbrace{\hat{x}(t)\sin\omega_c t}_{[-j\,\mathrm{sgn}(f)X(f)]*\left[\frac{1}{2}\delta(f-f_c)e^{-j\frac{\pi}{2}}+\frac{1}{2}\delta(f+f_c)e^{+j\frac{\pi}{2}}\right]}$$

$$\therefore Y(f) = \tfrac{1}{2}X(f-f_c)+\tfrac{1}{2}X(f+f_c)+\left[\tfrac{1}{2}X(f-f_c)e^{-j\pi}\mathrm{sgn}(f-f_c)+\tfrac{1}{2}X(f+f_c)\,\mathrm{sgn}(f+f_c)\right]$$

$$= \tfrac{1}{2}X(f-f_c)\left[1-\mathrm{sgn}(f-f_c)\right]+\tfrac{1}{2}X(f+f_c)\left[1+\mathrm{sgn}(f+f_c)\right]$$



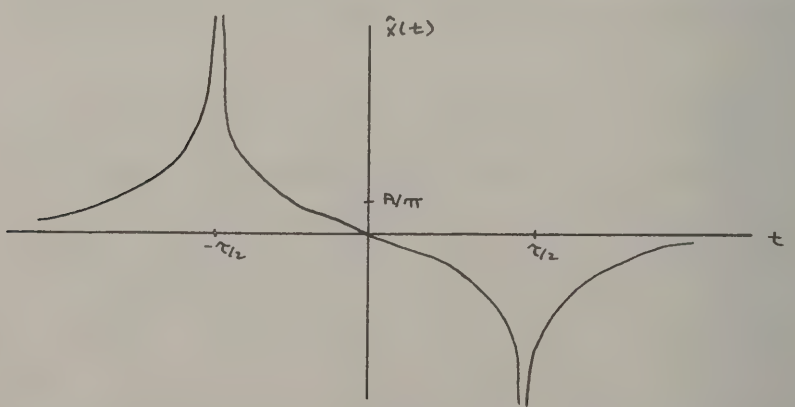
(2.82)

$$x(t) = A\,\Pi(t/\tau)$$

$$\hat{x}(t) = \int_{-\infty}^{\infty}\frac{x(t')}{\pi(t-t')}\,dt' = \frac{A}{\pi}\int_{-\tau/2}^{\tau/2}\frac{1}{t-t'}\,dt' = \frac{A}{\pi}\ln(t-t')\Big|_{t'=-\tau/2}^{t'=\tau/2}$$

$$= \frac{A}{\pi}\ln\left[\left|\frac{t-\tau/2}{t+\tau/2}\right|\right].$$

Note that the absolute value of the argument of ln comes from the symmetry of integrand about t.



Note that $\hat{x}(t)$ is odd, and doesn't resemble $x(t)$.

(2.83)

(a) $x(t) = \cos \omega_0 t$, $\hat{x}(t) = \sin \omega_0 t$.

$$\frac{1}{2T}\int_{-T}^{T} \hat{x}(t)\, x(t)\, dt = \frac{\omega_0}{2\pi}\int_0^{\omega_0/2\pi} \sin\omega_0 t \cos\omega_0 t = \frac{\omega_0}{4\pi}\int_0^{\omega_0/2\pi} \sin 2\omega_0 t\, dt = 0.$$

(b) $x(t) = 2\cos\omega_0 t + \sin\omega_0 t \cos^2 2\omega_0 t$

$= 2\cos\omega_0 t + \sin\omega_0 t [1/2 + 1/2 \cos 4\omega_0 t]$

$= 2\cos\omega_0 t + \frac{1}{2}\sin\omega_0 t + \frac{1}{4}\sin 5\omega_0 t - \frac{1}{4}\sin 3\omega_0 t$

$\hat{x}(t) = 2\sin\omega_0 t - \frac{1}{2}\cos\omega_0 t - \frac{1}{4}\cos 5\omega_0 t + \frac{1}{4}\cos 3\omega_0 t$

$x(t)\,\hat{x}(t) = -\cos^2\omega_0 t + \sin^2\omega_0 t + \Sigma$ odd functions

$$\therefore \frac{1}{T_0}\int_0^{T_0} x(t)\,\hat{x}(t)\, dt = \frac{\omega_0}{2\pi}\int_0^{T_0} [-\tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega_0 t + \tfrac{1}{2} - \cos 2\omega_0 t]\, dt$$

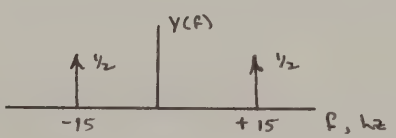
$$= 0.$$

(c) $$\int_{-\infty}^{\infty} \hat{x}(t)\, x(t)\, dt = \int_{-\tau/2}^{\tau/2} A\,\hat{x}(t)\, dt.$$

But from prob 2.66, $\hat{x}(t)$ is odd and so integrating over symmetrical limits gives zero.

(2.84)

(a)



(b) $y(t) = \cos 10\pi t \cos 40\pi t + \widehat{\cos 10\pi t}\, \sin 40\pi t$

$= \frac{1}{2}\cos 50\pi t + \frac{1}{2}\cos 30\pi t + \sin 10\pi t \sin 40\pi t$

$= \frac{1}{2}\cos 50\pi t + \frac{1}{2}\cos 30\pi t + \frac{1}{2}\cos 30\pi t - \frac{1}{2}\cos 50\pi t$

$= \cos 30\pi t.$

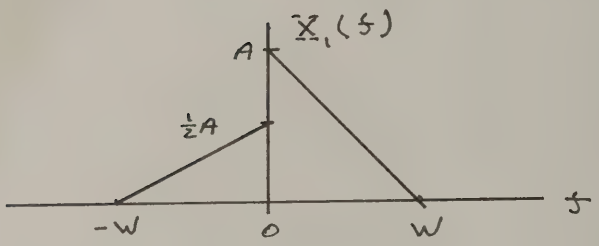
(2.85)

(a) $x_1(t) = \frac{3}{4}x(t) + \frac{1}{4}j\hat{x}(t)$

$$X_1(f) = \frac{3}{4}X(f) + \frac{1}{4}j(-j\,\mathrm{sgn}(f))X(f)$$

$$= X(f)\left[\frac{3}{4} + \frac{1}{4}\,\mathrm{sgn}(f)\right]$$

$$= \begin{cases} X(f), & f > 0 \\ \frac{1}{2}X(f), & f < 0 \end{cases}$$



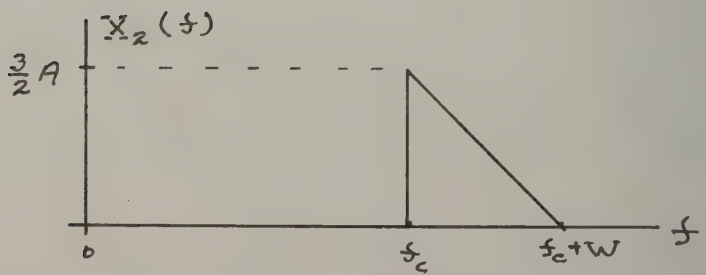
(b) $x_2(t) = \left[\frac{3}{4}x(t) + \frac{3}{4}j\hat{x}(t)\right]e^{j2\pi f_c t}$

where $f_c >> W$

$$X_2(f) = \frac{3}{4}X(f)\left[1 + \mathrm{sgn}(f)\right] * \delta(f - f_c)$$

$$= \frac{3}{4}X(f - f_c)\left[1 + \mathrm{sgn}(f - f_c)\right]$$

$$= \begin{cases} \frac{3}{2}X(f - f_c), & f > f_c \\ 0, & f < f_c \end{cases}$$



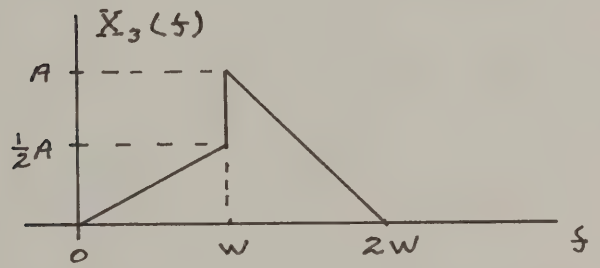
(2.85)

(c) $x_3(t) = \left[\frac{3}{4}x(t) + j\frac{1}{4}\hat{x}(t)\right]e^{j2\pi Wt}$

$$X_3(f) = X(f)\left[\frac{3}{4} + \frac{1}{4}\,\mathrm{sgn}(f)\right] * \delta(f-W)$$

$$= X(f-W)\left[\frac{3}{4} + \frac{1}{4}\,\mathrm{sgn}(f-W)\right]$$

$$= \begin{cases} X(f-W), & f > W \\ \frac{1}{2}X(f-W), & f < W \end{cases}$$

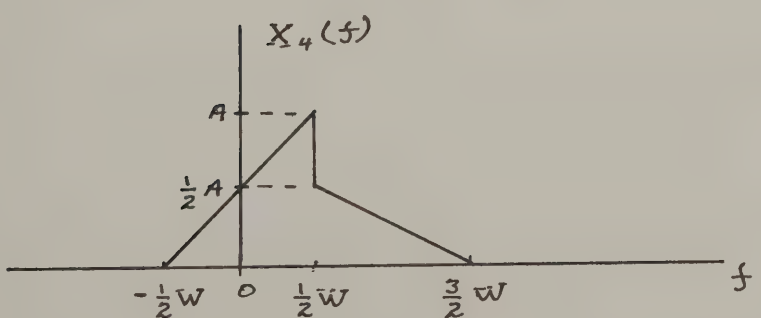


(d) $x_4(t) = \left[\frac{3}{4}x(t) - j\frac{1}{4}\hat{x}(t)\right]e^{j\pi Wt}$

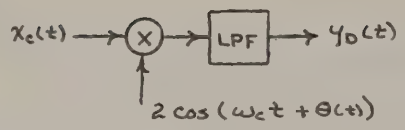
$$= X(f)\left[\frac{3}{4} - \frac{1}{4}\,\mathrm{sgn}(f)\right] * \delta\left(f-\frac{W}{2}\right)$$

$$= X\left(f-\tfrac{1}{2}W\right)\left[\frac{3}{4} - \frac{1}{4}\,\mathrm{sgn}\left(f-\tfrac{1}{2}W\right)\right]$$

$$= \begin{cases} \frac{1}{2}X\left(f-\frac{1}{2}W\right), & f > \frac{1}{2}W \\ X\left(f-\frac{1}{2}W\right), & f < \frac{1}{2}W \end{cases}$$



(3.1)

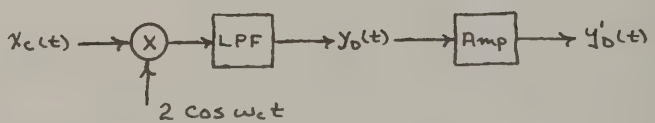


$x_c(t) = m(t) \sin[\omega_c t + \phi(t)]$.

$y_D(t) = m(t) \sin[\omega_c t + \phi(t)]\, 2\cos[\omega_c t + \theta(t)]$

$= m(t) \underbrace{\sin[\phi(t) - \theta(t)]}_{\text{Ideally, this is a non-zero constant. If } \phi(t)-\theta(t) = \pi/2\text{, then it equals 1.}} + \underbrace{m(t)\cos[2\omega_c t + \phi(t) + \theta(t)]}_{\text{filtered out}}$

(3.2)



$x_c(t) = A'_c [1 + a\, m_n(t)] \cos\omega_c t$

$y_D(t) = \{A'_c [1 + a\, m_n(t)] \cos\omega_c t\}\, 2\cos\omega_c t$

$= \underbrace{A'_c}_{\text{No DC coupling removes this term.}} + A'_c a\, m_n(t) + \underbrace{[A'_c + A'_c a\, m_n(t)] \cos 2\omega_c t}_{\text{filtered out}}$

$\therefore\ y_D(t) = A'_c a\, m_n(t)$.

The amplifier that follows $y_D(t)$ can scale to give $y'_D(t) = m(t)$.

(3.3)

$x_c(t) = A_c[1 + a\,m_n(t)]\cos \omega_c t.$

Efficiency $= \dfrac{a^2\langle m_n^2(t)\rangle}{1 + a^2\langle m_n^2(t)\rangle} \times 100\%$, where a is the modulation index, and $m_n(t)$ is the message signal normalized so that its maximum value is unity.

For square-wave:

$\langle m_n^2(t)\rangle = 1$. Efficiency $= \dfrac{100a^2}{1+a^2}\ \%$.

For triangular-wave:

$$\langle m_n^2(t)\rangle = \frac{1}{2T}\int_{-T}^{T} m_n^2(t)\,dt = \frac{4}{T}\int_0^{T/4} m_n^2(t)\,dt = \frac{64}{T^3}\int_0^{T/4} t^2\,dt = 1/3.$$

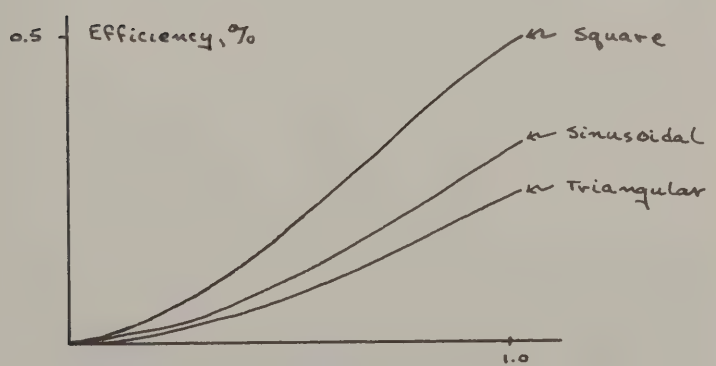
$$\text{Efficiency} = \frac{100a^2}{3+a^2}\ \%.$$

For sinusoidal-wave:

$$\langle m_n^2(t)\rangle = \frac{1}{2T}\int_{-T}^{T} \sin^2(2\pi t/T)\,dt = \frac{2}{T}\int_0^{T/2} \sin^2(2\pi t/T)\,dt$$

$$= \frac{1}{T}\int_0^{T/2} [1 + \cos(4\pi t/T)]\,dt = 1/2.$$

$$\text{Efficiency} = \frac{100a^2}{2+a^2}\ \%.$$



(3.4) From the previous problem $\langle m_n^2(t) \rangle = 1/3$

From the figure: Max value of envelope = 6

Min value of envelope = 2

Assume $|\text{Max } m(t)| = |\text{Min } m(t)|$

The envelope of $x_c(t)$ is $A_c'[1 + a m_n(t)]$

Thus

$$A_c'(1+a) = 6$$
$$A_c'(1-a) = 2$$

$$\frac{1+a}{1-a} = 3 \qquad 1 + a = 3 - 3a$$
$$4a = 2$$
$$a = 1/2$$
$$A_c'(3/2) = 6$$
$$A_c' = 6\left(\tfrac{2}{3}\right) = 4$$

The carrier power is $\frac{1}{2}(A_c')^2 = 8$ Watts

$$\text{Efficiency} = \frac{a^2 \langle m_n^2(t) \rangle}{1 + a^2 \langle m_n^2(t) \rangle} = \frac{\frac{1}{4}\frac{1}{3}}{1 + \frac{1}{4}\frac{1}{3}} = \frac{1}{13}$$

Efficiency = 0.0769 = 7.69%

(3.5) $m(t) = -6 \cos 20\pi t - 2 \cos 60\pi t$

(a) $|\text{min } m(t)| = |-8| = 8$

Thus

$$m_n(t) = \tfrac{1}{8}(-6 \cos 20\pi t - 2 \cos 60\pi t)$$
$$= -\tfrac{3}{4} \cos 20\pi t - \tfrac{1}{4} \cos 60\pi t$$

(b) $\langle m_n^2(t) \rangle = \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{10}{32} = \frac{5}{16}$

(c) $$E = \frac{a^2 \langle m_n^2(t) \rangle}{1 + a^2 \langle m_n^2(t) \rangle} = \frac{\frac{1}{4}\frac{5}{16}}{1 + \frac{1}{4}\frac{5}{16}}$$

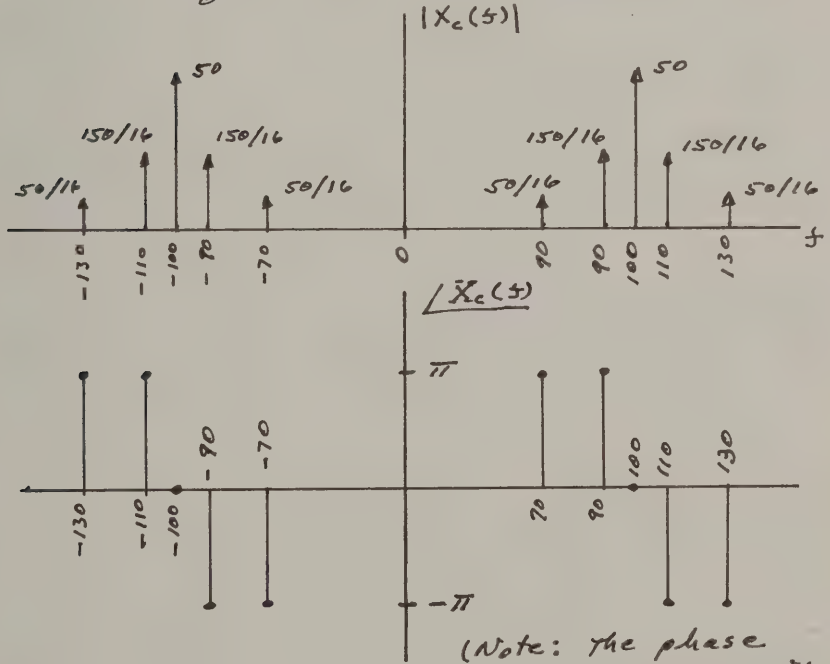
$$= \frac{5}{64+5} = \frac{5}{69} = 0.0725 = 7.25\%$$

(3.5) (Cont.)

$$x_c(t) = 100\left[1 + \tfrac{1}{2}\left(-\tfrac{3}{4}\cos 20\pi t - \tfrac{1}{4}\cos 60\pi t\right)\right] \cdot \cos 200\pi t$$

$$x_c(t) = 100\cos 200\pi t - 50\left(\tfrac{3}{4}\cos 20\pi t + \tfrac{1}{4}\cos 60\pi t\right)\cos 200\pi t$$

$$x_c(t) = 100\cos 200\pi t - \tfrac{150}{8}\cos 220\pi t - \tfrac{150}{8}\cos 180\pi t - \tfrac{50}{8}\cos 260\pi t - \tfrac{50}{8}\cos 140\pi t$$



(Note: the phase spectrum is not unique)

(3.6) $m(t) = 6\cos 20\pi t + 2\cos 60\pi t$

The key to this problem is to note that $m(t)$ consists of a fundamental (10 Hz) and a third harmonic (30 Hz)

(3.6) (cont)

Thus $m(t)$ consists of only odd harmonics. Thus $m(t)$ has half-wave symmetry. Therefore the maximum value of $m(t)$ is $+8$ and the minimum value of $m(t)$ is -8.

(a) $m_n(t) = \frac{1}{8}(6\cos 20\pi t + 2\cos 60\pi t)$

$= \frac{3}{4}\cos 20\pi t + \frac{1}{4}\cos 60\pi t$

(b) $\langle m_n^2(t)\rangle = \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{5}{16}$

(c) $E = \dfrac{a^2\langle m_n^2(t)\rangle}{1 + a^2\langle m_n^2(t)\rangle} = \dfrac{\frac{1}{4}\left(\frac{5}{16}\right)}{1 + \frac{1}{4}\left(\frac{5}{16}\right)}$

$= 7.25\%$

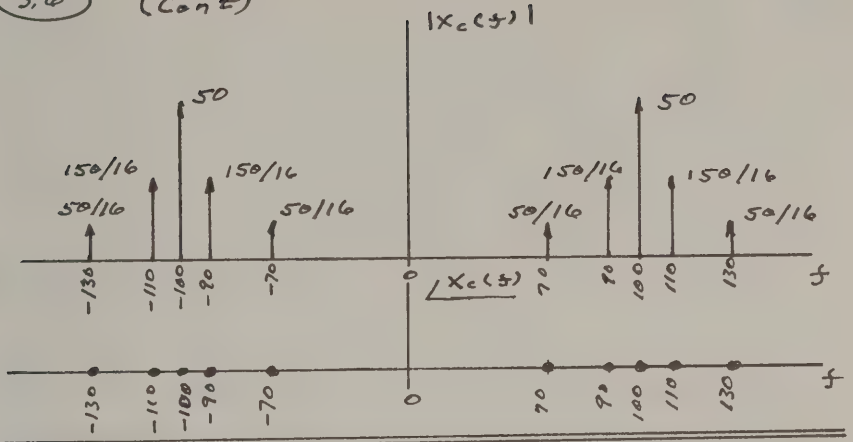
(d) The spectrum of $x_c(t)$ follows from the previous problem. Only the sign of $m(t)$ is changed. Thus

$x_c(t) = 100\cos 200\pi t$

$+ \frac{150}{8}\cos 220\pi t + \frac{150}{8}\cos 180\pi t$

$+ \frac{50}{8}\cos 260\pi t + \frac{50}{8}\cos 140\pi t$

(3.6) (Cont)



(3.7) $x_c(t) = 20 \cos 2\pi(150)t + 6\cos(160)t + 6\cos 2\pi(140)t$

Since the spectrum is symmetrical about 150 Hertz, the carrier frequency must be 150 Hz. First write -

$$x_c(t) = 20\cos 2\pi(150)t + 12\cos 2\pi(10)t \cos 2\pi(150)t$$
$$= 20\left[1 + \frac{12}{20}\cos 2\pi(10)t\right]\cos 2\pi(150)t$$

Thus

$$a = \frac{12}{20} = 0.6$$

$$m_n(t) = \cos 2\pi(10)t$$

$$\langle m_n^2(t)\rangle = \frac{1}{2}$$

$$E = \frac{a^2\langle m_n^2(t)\rangle}{1 + a^2\langle m_n^2(t)\rangle} = \frac{(0.6)^2(\frac{1}{2})}{1 + (0.6)^2(\frac{1}{2})}$$

$$= \frac{9}{59} = 15.25\%$$

3.8

$$m(t) = 3\cos(2\pi f_m t) + \cos(4\pi f_m t) + 2\cos(10\pi f_m t)$$

If $m(t)$ is plotted, it is seen that $\min m(t) = -4$ and occurs at $f_m t = 0.5$

Thus $|\min m(t)| = 4$ and

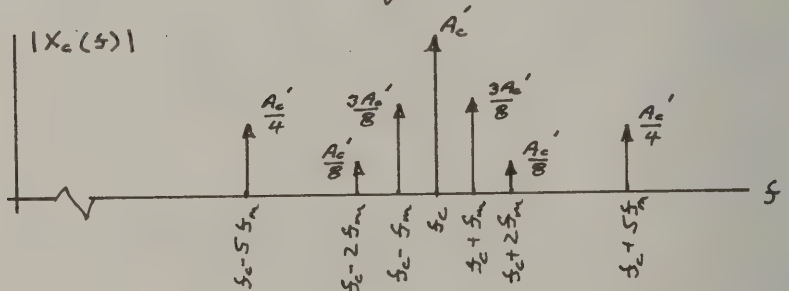
$$m_n(t) = \frac{3}{4}\cos(2\pi f_m t) + \frac{1}{4}\cos(4\pi f_m t) + \frac{1}{2}\cos(10\pi f_m t)$$

Since $a = 1$, $x_c(t)$ is given by

$$x_c(t) = A_c'\left[1 + \frac{3}{4}\cos(2\pi f_m t) + \frac{1}{4}\cos(4\pi f_m t) + \frac{1}{2}\cos(10\pi f_m t)\right]\cos 2\pi f_c t$$

$$x_c(t) = A_c'\cos 2\pi f_c t + \frac{3}{8}A_c'\cos 2\pi(f_c+f_m)t + \frac{3}{8}A_c'\cos 2\pi(f_c-f_m)t + \frac{1}{8}A_c'\cos 2\pi(f_c+2f_m)t + \frac{1}{8}A_c'\cos 2\pi(f_c-2f_m)t + \frac{1}{4}A_c'\cos 2\pi(f_c+5f_m)t + \frac{1}{4}A_c'\cos 2\pi(f_c-5f_m)t$$

We shall draw the single-sided spectrum



$$\langle m_n^2 \rangle = \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2$$

$$= \frac{9}{32} + \frac{1}{32} + \frac{4}{32} = \frac{14}{32} = \frac{7}{16}$$

$$\text{Eff} = \frac{7/16}{1 + 7/16} = \frac{7}{23} = 30.43\%$$

(3.9)

(a) $x(t) = m(t) + \cos \omega_c t$

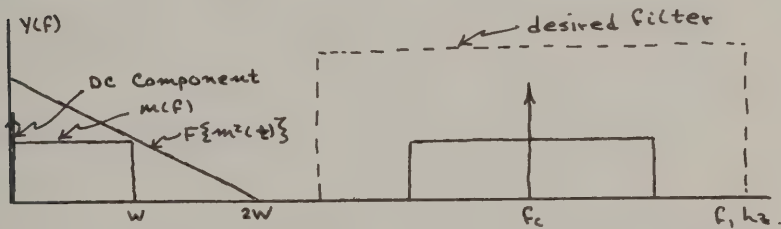
$y(t) = 2x(t) + 4x^2(t) = 2[m(t) + \cos\omega_c t] + 4[m(t) + \cos\omega_c t]^2$

$= 2m(t) + 2\cos\omega_c t + 4m^2(t) + 8m(t)\cos\omega_c t + 4\cos^2\omega_c t$

$= 2 + 2m(t) + 4m^2(t) + 2\cos 2\omega_c t + 8m(t)\cos\omega_c t + 2\cos\omega_c t$

(b) Note that if bandwidth of $m(t)$ is W, then bandwidth of $m^2(t)$ is $2W$ because

$$F\{m^2(t)\} = F\{m(t)\} * F\{m(t)\}.$$



(Center freq. of filter = f_c, Bandwidth $\geq 2W$)

(c) $g(t) = 2\cos\omega_c t + 8m(t)\cos\omega_c t = 8[\tfrac{1}{4} + m(t)]\cos\omega_c t$

For mod. index = 0.1, $\frac{M}{1/4} = 0.1 \Rightarrow M = 0.025$.

(d) With this implementation, a high quality multiplier is no longer needed.

(3.10)

The required figure is on the following page.

For lower sideband $H_L(f) = \frac{1}{2}[\text{sgn}(f+f_c) - \text{sgn}(f-f_c)]$

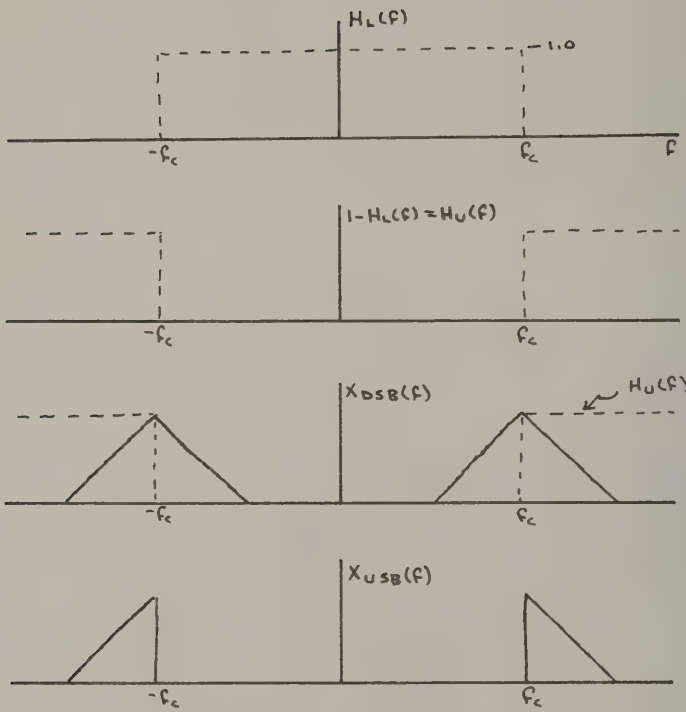
$\therefore H_U(f) = 1 - H_L(f) = 1 + \frac{1}{2}\text{sgn}(f-f_c) - \frac{1}{2}\text{sgn}(f+f_c)$.

$X_{USB}(f) = H_U(f)\,X_{DSB}(f) = [1 + \frac{1}{2}\text{sgn}(f-f_c) - \frac{1}{2}\text{sgn}(f+f_c)][\frac{1}{2}A_c M(f+f_c) + \frac{1}{2}A_c M(f-f_c)]$

$= \frac{1}{4}A_c[M(f+f_c) + M(f-f_c)] - \underbrace{\frac{1}{4}A_c[M(f+f_c)\text{sgn}(f+f_c) - M(f-f_c)\text{sgn}(f-f_c)]}_{F\{\frac{1}{2}A_c\hat{m}(t)\sin\omega_c t\}}$

$\therefore X_{USB}(t) = \frac{1}{2}A_c m(t)\cos\omega_c t - \frac{1}{2}A_c\hat{m}(t)\sin\omega_c t$

(3.10) (Cont.)



(3.11)

Define the output of summer as $y(t)$. Then

$$y(t) = 2X_r(t)\cos\omega_c t - 2X_r(t)\sin\omega_c t$$

$$= [A_c m(t)\cos\omega_c t + A_c \hat{m}(t)\sin\omega_c t]\cos\omega_c t - [A_c m(t)\cos\omega_c t + A_c \hat{m}(t)\sin\omega_c t]\sin\omega_c t$$

$$= \tfrac{1}{2}A_c m(t) + \tfrac{1}{2}A_c m(t)\cos 2\omega_c t + \tfrac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t - A_c m(t)\sin 2\omega_c t - \underbrace{\tfrac{1}{2}A_c \hat{m}(t) + \tfrac{1}{2}A_c \hat{m}(t)\cos 2\omega_c t}_{\frac{1}{2}A_c m(t)}$$

$\therefore$ $y(t) = A_c m(t)$ + High freq. terms.

And so $y_D(t) = A_c m(t)$.

(3.12)

$$y(t) = 2x_r(t)\cos\omega_c t + \overbrace{2x_r(t)\sin\omega_c t}$$

$$= [A_c m(t)\cos\omega_c t + A_c \hat{m}(t)\sin\omega_c t]\cos\omega_c t + \overbrace{[A_c m(t)\cos\omega_c t + A_c \hat{m}(t)\sin\omega_c t]\sin\omega_c t}$$

$$= \tfrac{1}{2}A_c m(t) + \tfrac{1}{2}A_c m(t)\cos 2\omega_c t + \tfrac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t + \underbrace{\tfrac{1}{2}\overbrace{A_c m(t)\sin 2\omega_c t}}_{-\frac{1}{2}A_c m(t)\cos 2\omega_c t} + \underbrace{\tfrac{1}{2}\overbrace{A_c \hat{m}(t)}}_{-\frac{1}{2}A_c m(t)} - \underbrace{\tfrac{1}{2}\overbrace{A_c \hat{m}(t)\cos 2\omega_c t}}_{\frac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t}$$

$$= A_c m(t) \Rightarrow \text{No filtering required.}$$

Note: The purpose of Probs. 3.13 and 3.14 is to give additional practice in manipulating Hilbert Transforms.

(3.13)

$$y(t) = 2x_r(t)\cos\omega_c t - \overbrace{2x_r(t)\sin\omega_c t}$$

$$= [A_c m(t)\cos\omega_c t - A_c \hat{m}(t)\sin\omega_c t]\cos\omega_c t - \overbrace{[A_c m(t)\cos\omega_c t - A_c \hat{m}(t)\sin\omega_c t]\sin\omega_c t}$$

$$= \tfrac{1}{2}A_c m(t) + \tfrac{1}{2}A_c m(t)\cos 2\omega_c t - \tfrac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t - \overbrace{\tfrac{1}{2}A_c m(t)\sin 2\omega_c t} + \overbrace{\tfrac{1}{2}A_c \hat{m}(t)} - \overbrace{\tfrac{1}{2}A_c \hat{m}(t)\cos 2\omega_c t}$$

$$= A_c m(t)\cos 2\omega_c t - A_c \hat{m}(t)\sin 2\omega_c t.$$

Since $y(t)$ contains no baseband components, $y_0(t)$ is identically zero.

Note that we haven't demodulated, but still have an USB signal centered at a carrier freq. of $2f_c$.

(3.14)

$$y(t) = 2x_r(t)\cos\omega_c t + \overbrace{2x_r(t)\sin\omega_c t}, \text{ where}$$

$$x_r(t) = \tfrac{1}{2}A_c m(t)\cos\omega_c t + \tfrac{1}{2}A_c \hat{m}(t)\sin\omega_c t.$$

$$\therefore y(t) = \tfrac{1}{2}A_c m(t) + \tfrac{1}{2}A_c m(t)\cos 2\omega_c t + \tfrac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t$$

$$+ \overbrace{\tfrac{1}{2}A_c m(t)\sin 2\omega_c t} + \overbrace{\tfrac{1}{2}A_c \hat{m}(t)} - \overbrace{\tfrac{1}{2}A_c \hat{m}(t)\cos 2\omega_c t}$$

$$-\tfrac{1}{2}A_c m(t)\cos 2\omega_c t \qquad -\tfrac{1}{2}A_c m(t) \qquad \tfrac{1}{2}A_c \hat{m}(t)\sin 2\omega_c t$$

$$\therefore y_0(t) = y(t) = 0.$$

(3.15)

The message signal $m(t)$ is assumed a power signal. The power in $A_c m(t)\cos\omega_c t$ is $\langle A_c^2 m^2(t)\cos^2\omega_c t\rangle$. If $m(t)$ is lowpass with respect to ω_c then

$$\langle A_c^2 m^2(t)\cos^2\omega_c t\rangle = \tfrac{1}{2}A_c^2\langle m^2(t)\rangle + \tfrac{1}{2}A_c^2\langle m^2(t)\rangle\langle\cos 2\omega_c t\rangle$$

$$= \tfrac{1}{2}A_c^2\langle m^2(t)\rangle.$$

Filtering this signal with an ideal sideband filter removes $\frac{1}{2}$ of the signal spectrum which is symmetric about the carrier, and thus reduces the power by $\frac{1}{2}$. Thus, the power in the SSB signal is $\frac{1}{4}A_c^2\langle m^2(t)\rangle$.

Next we find the time-average power of the phase shift modulated signal.

$$P_{Avg.} = \langle[\tfrac{1}{2}A_c m(t)\cos\omega_c t \pm \tfrac{1}{2}A_c \hat{m}(t)\sin\omega_c t]^2\rangle$$

$$= \tfrac{1}{4}A_c^2\langle m^2(t)\cos^2\omega_c t\rangle + \tfrac{1}{4}A_c^2\langle \hat{m}^2(t)\sin^2\omega_c t\rangle$$

$$\pm \tfrac{1}{2}A_c^2\langle m(t)\hat{m}(t)\sin\omega_c t\cos\omega_c t\rangle$$

$$= \tfrac{1}{8}A_c^2\langle m^2(t)\rangle + \tfrac{1}{8}A_c^2\langle \hat{m}^2(t)\rangle \pm \tfrac{1}{4}A_c^2\langle m(t)\hat{m}(t)\rangle$$

$$= \tfrac{1}{4}A_c^2\langle m^2(t)\rangle.$$

(3.16)

$x_{AM}(t) = A_c[A + m(t)]$ $\qquad x_c(t) = \begin{cases} x_{AM}(t)\cos\omega_c t \\ \text{or} \\ x_{DSB}(t)\cos\omega_c t \end{cases}$

$x_{DSB}(t) = A_c m(t)$

Model switch as $\sum_{N=-\infty}^{\infty} C_n e^{jn\omega_c t}$ where we may think of switch as unity periodic pulse train of any duty cycle desired.

Switch output $= y(t) = x_c(t) \sum_{N=-\infty}^{\infty} C_n e^{jn\omega_c t}$

For DSB:

$$y(t) = x_{DSB}(t)\cos\omega_c t \sum_{N=-\infty}^{\infty} C_n e^{jn\omega_c t}$$

$$= x_{DSB}(t)\left[C_{1/2}\, e^{j\omega_c t} + C_{-1/2} e^{-j\omega_c t}\right] \sum_{N=-\infty}^{\infty} C_n e^{jn\omega_c t}$$

$$= x_{DSB}(t)\left[\tfrac{C_1}{2} e^{j2\omega_c t} + C + \tfrac{C_{-1}}{2} e^{-j2\omega_c t}\right] + x_{DSB}(t) \sum_{N=-\infty}^{\infty} C_n e^{jn\omega_c t}$$

where $C = \frac{1}{2}(C_1 + C_{-1})$. For $C=1$ we have $N \neq \pm 1$

Output of filter = low pass portion of $y(t) = LP\{y(t)\}$.

$\therefore\ LP\{y(t)\} = x_{DSB}(t) = A_c m(t)$.

For AM:

As above, but $LP\{y(t)\} = x_{AM}(t) = A_c[A + m(t)]$ which for no DC coupling gives $A_c m(t)$.

(3.17)

The DSB signal is given by

$$m(t)c(t) = (A_m \cos\omega_m t)(A_c \sin\omega_c t)$$

$$= \tfrac{1}{2} A_m A_c \sin(\omega_c - \omega_m)t + \tfrac{1}{2} A_m A_c \sin(\omega_c + \omega_m)t$$

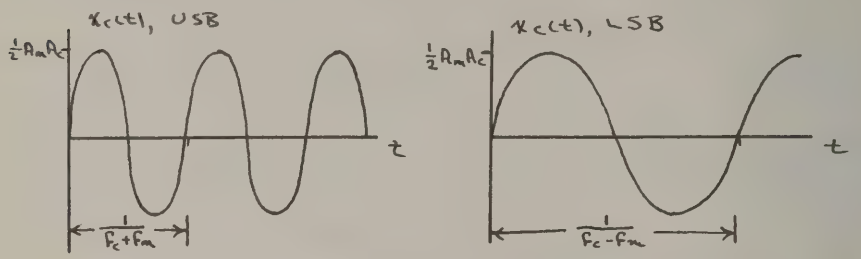
Thus for USB: $x_c(t) = \frac{1}{2} A_m A_c \sin(\omega_c + \omega_m)t$

and for LSB: $x_c(t) = \frac{1}{2} A_m A_c \sin(\omega_c - \omega_m)t$.

Note that (3.23) and (3.24) <u>cannot</u> be used since the derivation assumed that the

(3.17) (Cont.)

DSB signal was formed using a carrier of the form $c(t) = K \cos \omega_c t$. Plots of $x_c(t)$ are shown below.



(3.18)

$$x_c(t) = \frac{A\epsilon}{2}\cos[(\omega_c-\omega_1)t+\phi] + \frac{A(1-\epsilon)}{2}\cos[(\omega_c+\omega_1)t+\theta_1] + \frac{B}{2}\cos[(\omega_c+\omega_2)t+\theta_2].$$

Multiplying $x_c(t)$ by $4\cos\omega_c t$ gives

$$y(t) = A\epsilon\cos[(2\omega_c-\omega_1)t+\phi] + A\epsilon\cos[\omega_1 t-\phi] + A(1-\epsilon)\cos[(2\omega_c+\omega_1)t+\theta_1] + A(1-\epsilon)\cos[\omega_1 t+\theta_1] + B\cos[(2\omega_c+\omega_2)t+\theta_2] + B\cos[\omega_2 t+\theta_2].$$

Lowpass portion of $y(t) = LP\{y(t)\}$.

$$\therefore LP\{y(t)\} = \underbrace{A\epsilon\cos(\omega_1 t-\phi)}_{\#1} + \underbrace{A(1-\epsilon)\cos(\omega_1 t+\theta_1)}_{\#2} + \underbrace{B\cos(\omega_2 t+\theta_2)}_{\#3}.$$

Noting that amplitude of term #3 is the same as baseband, terms #1 and #2 must add so that at ω_1 we have amplitude A. Thus $\theta_1 = -\phi$. This gives

$$LP\{y(t)\} = A\cos(\omega_1 t+\theta_1) + B\cos(\omega_2 t+\theta_2)$$
$$= A\cos\omega_1\left(t+\frac{\theta_1}{\omega_1}\right) + B\cos\omega_2\left(t+\frac{\theta_2}{\omega_2}\right).$$

(3.18) (Cont.)

For no distortion we require $\frac{\theta_1}{\omega_1} = \frac{\theta_2}{\omega_2}$.

Thus $\theta_2 = \frac{\omega_2}{\omega_1}\theta_1 = -\frac{\omega_2}{\omega_1}\phi$.

In general, phase must be linear and have odd symmetry about the carrier frequency.

(3.19)

We shall assume $m(t)$ of the form

$$m(t) = A\cos\omega_1 t + B\cos\omega_2 t.$$

Then $x_c(t)$, the VSB modulated waveform is

$$x_c(t) = \tfrac{1}{2}A\epsilon\cos(\omega_1 t) + \tfrac{1}{2}A(1-\epsilon)\cos(\omega_c+\omega_1)t + \tfrac{1}{2}B\cos(\omega_c+\omega_2)t.$$

Let $y(t)$ = received waveform + carrier = $x_c(t) + K\cos\omega_c t$.

Thus,

$$y(t) = \frac{A\epsilon}{2}[\cos\omega_c t\cos\omega_1 t + \sin\omega_c t\sin\omega_1 t] + \frac{A(1-\epsilon)}{2}[\cos\omega_c t\cos\omega_1 t - \sin\omega_c t\sin\omega_1 t] + \frac{B}{2}[\cos\omega_c t\cos\omega_2 t - \sin\omega_c t\sin\omega_2 t] + K\cos\omega_c t.$$

$$\therefore y(t) = y_1(t)\cos\omega_c t + y_2(t)\sin\omega_c t$$

where $y_1(t) = \frac{A}{2}\cos\omega_1 t + \frac{B}{2}\cos\omega_2 t + K$

$$y_2(t) = (A\epsilon - \tfrac{A}{2})\sin\omega_1 t - \tfrac{B}{2}\sin\omega_2 t.$$

$y(t) = R(t)\cos(\omega_c t + \theta)$ where $R(t) = \sqrt{y_1^2(t) + y_2^2(t)}$

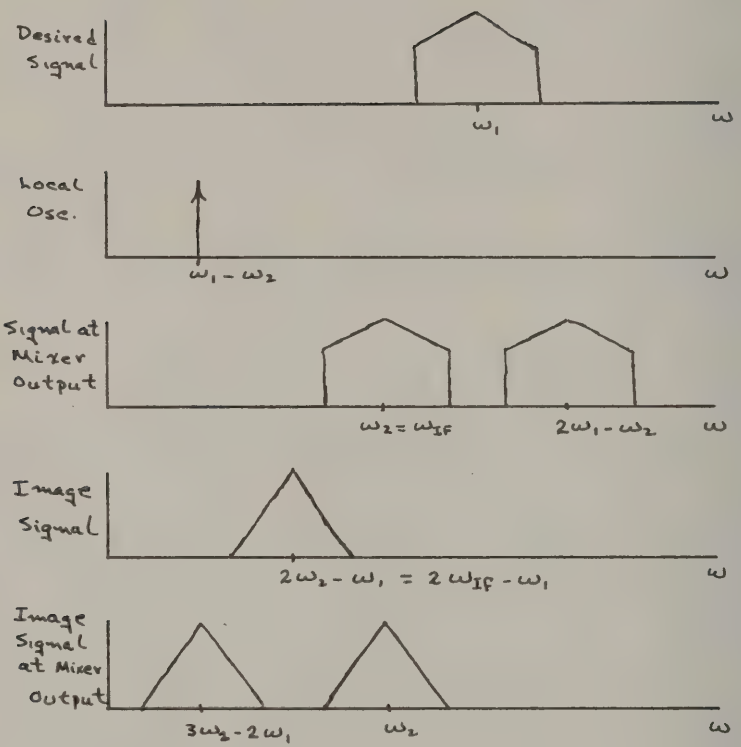
The envelope detector output is $R(t)$.

For K large $R(t) \approx |y_1(t)|$.

$\therefore y_0(t) \approx \frac{A}{2}\cos\omega_1 t + \frac{B}{2}\cos\omega_2 t + K_1$

and for no DC coupling $y_0(t)$ is a replica of transmitted signal.

(3.20)

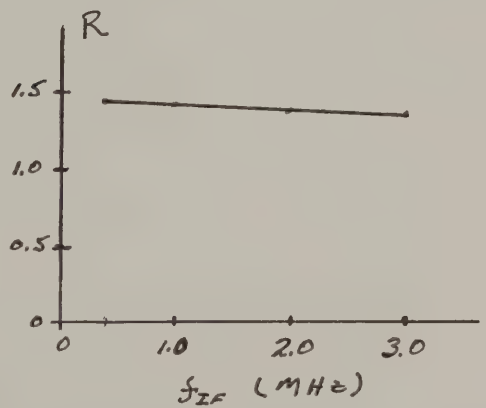

Desired Signal
ω_1
ω
Local Osc.
$\omega_1 - \omega_2$
ω
Signal at Mixer Output
$\omega_2 = \omega_{IF}$
$2\omega_1 - \omega_2$
ω
Image Signal
$2\omega_2 - \omega_1 = 2\omega_{IF} - \omega_1$
ω
Image Signal at Mixer Output
$3\omega_2 - 2\omega_1$
ω_2
ω

(3.21) Since high-side tuning is used the local oscillator frequency is $f_i + f_{IF}$, where f_i is the input frequency. Thus the ratio is

$$R = \frac{9 + f_{IF}}{6 + f_{IF}}$$

where f_{IF} is the IF frequency in MHz.

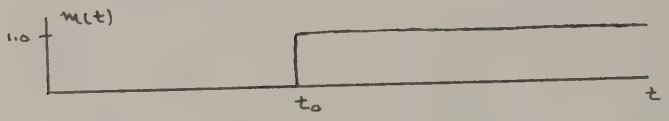
f_{IF}	R
0.4	1.47
0.6	1.45
1.0	1.43
2.0	1.38
3.0	1.33

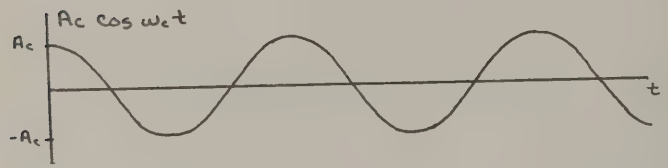


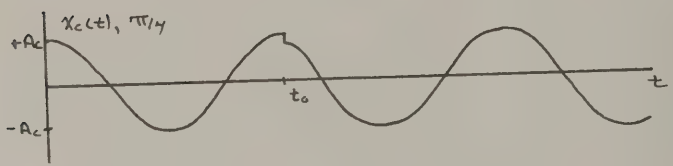
(3.22) From (3.50) and (3.51) the following table is constructed

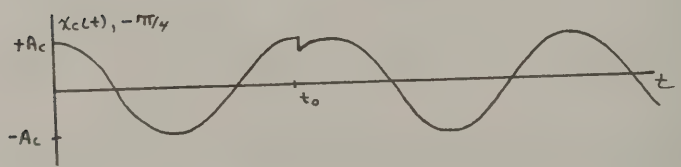
Input Frequency f_i	Local Oscillator $f_{LO} = f_i + f_{IF}$	Image Frequency $f_I = f_i \pm 2f_{IF}$
1490	1940	2400
1490	1035	580

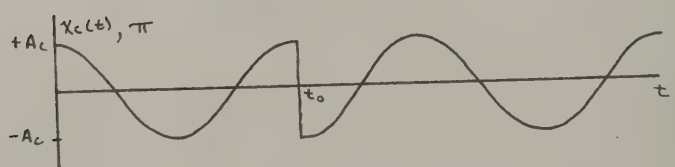
(3.23) $x_c(t) = A_c \cos [\omega_c t + K_p m(t)]$.

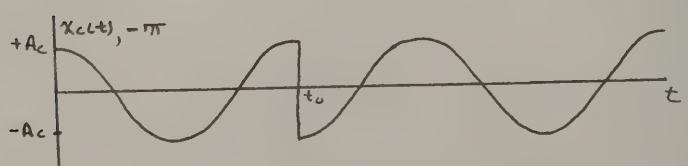












(3.24)

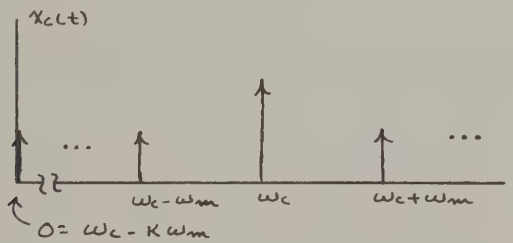
Consider $x_c(t) = \cos(\omega_c t + \beta \sin \omega_m t)$. This can be expressed as the Fourier Series

$$x_c(t) = \sum_{N=-\infty}^{\infty} J_N(\beta) \cos(\omega_c t + N\omega_m t).$$

Several cases of interest arise:

1. There is an integer K such that $\omega_c = K\omega_m$.
   a. β is large so that $|J_K(\beta)| > 0$.
   b. β is small so that $|J_K(\beta)| \approx 0$.
2. There is no K such that $\omega_c = K\omega_m$.

For Case 1, the magnitude spectrum of $x_c(t)$ is



For Case 1a there is a component at DC but for Case 1b this DC component is zero. For Case 2 there will be no component at zero frequency.

Note that K must be such that the corresponding negative portion appearing at zero frequency aids the positive portion. That is, $J_{-K}(\beta) = J_K(\beta)$. Thus, K is even.

(3.25)

$x_c(t) = A_c \cos\left[\omega_c t + K_f \int_{-2}^{t} \cos 8\pi t \, dt\right]$.

Peak frequency deviation = $f_d \max[m(t)]$.

β = Peak frequency deviation $\div f_m = \frac{32}{4} = 8$.

Only the components at f_c, $f_c \pm 4$ hz, $f_c \pm 8$ hz are passed.

$\therefore P_r = J_0^2(8) + 2\sum_{n=1}^{2} J_n^2(8)$

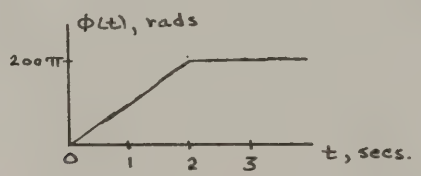
$= (0.172)^2 + 2(0.235)^2 + 2(0.113)^2 = 0.166$

Power out of filter = $100 P_r = 16.57$ watts.

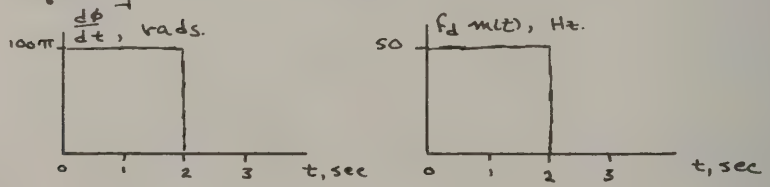
(3.26)

(a) $\phi(t) = 2\pi f_d \int_0^t 5\pi\left(\frac{\alpha-1}{2}\right) d\alpha$ where $f_d = 10$

$$= \begin{cases} 100\pi t & \text{for } 0 \le t \le 2 \\ 200\pi & \text{for } 2 < t \end{cases}$$



(b) Frequency deviation = $\frac{d\phi}{dt} = 100\pi\left(\frac{t-1}{2}\right)$.



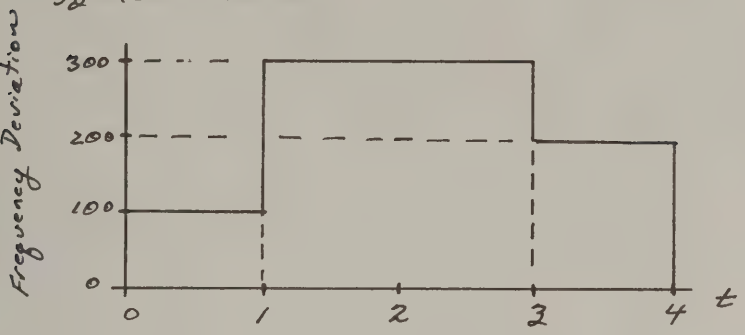
(C) From plots: Peak freq. Deviation = 50 hz.

(D) Peak Phase Deviation = 200π rads.

(E) Average Power is independent of mod. = $\frac{1}{2}A_c^2 = 800$ W.

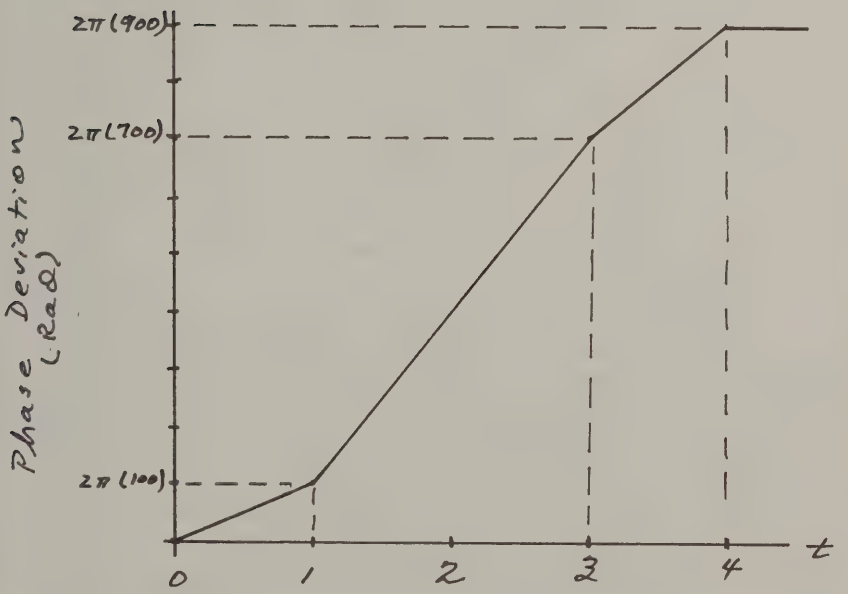
(3.27) The deviation in Hertz is

$f_\Delta m(t) = 10\, m(t)$



The phase deviation is

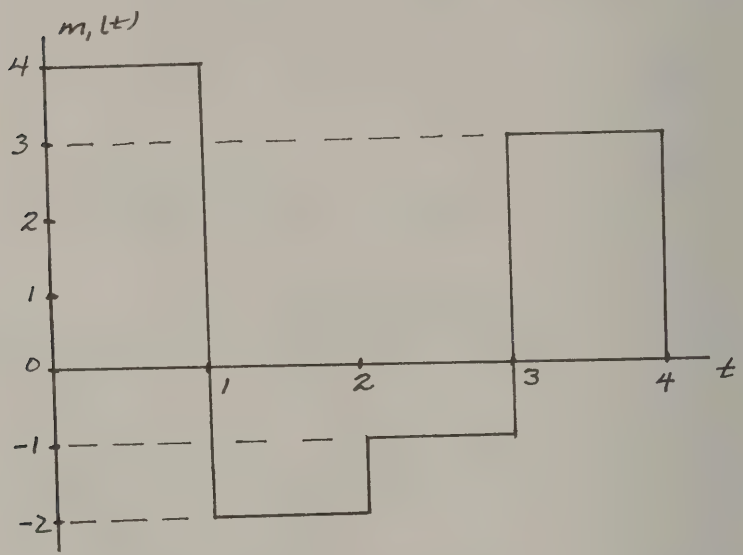
$$\phi(t) = 2\pi f_\Delta \int^t m(\alpha)\, d\alpha = 20\pi \int^t m(\alpha)\, d\alpha$$



(3.28) Since $\phi(t) = 2\pi f_\Delta \int^t m(\alpha)\, d\alpha$

Thus $m(t) = \frac{1}{2\pi f_\Delta} \frac{d}{dt}\phi(t)$

and $m_1(t) = \frac{1}{10\pi} \frac{d}{dt}\phi_1(t)$



$m_1(t) = 4, \quad 0 < t < 1$

$m_1(t) = -2, \quad 1 < t < 2$

$m_1(t) = -1, \quad 2 < t < 3$

$m_1(t) = 3, \quad 3 < t < 4$

$m_1(t) = 0, \quad t < 0 \text{ or } t > 4$

Since $m_2(t) = \frac{1}{10\pi} \frac{d}{dt}\phi_2(t)$ we have:

For $0 < t < 1$

$\phi_2(t) = K_1 t^2$

$\phi_2(1) = K_1 = 20\pi$

$\phi_2(t) = 20\pi t^2$

$m_2(t) = \frac{1}{10\pi} 20\pi (2t)$

$= 4t$

(3.28) (Cont.)

For $1 \le t \le 2$ $\quad m_2(t) = \frac{1}{10\pi}(-10\pi)$

$= -1$

For $2 \le t \le 3$ $\quad \phi_2(t) = k_2(3-t)^2$

$\phi_2(t) = k_2(3-2)^2 = 10\pi$

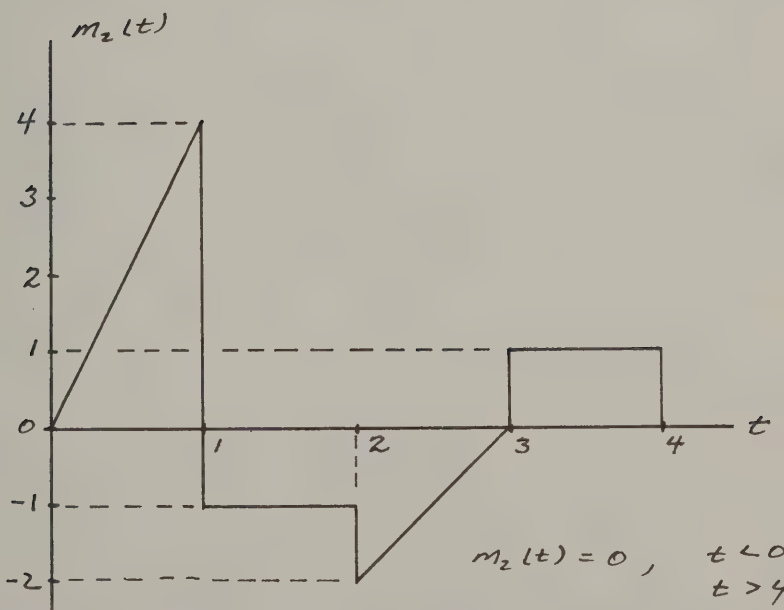
$\therefore k_2 = 10\pi$

$\phi_2(t) = 10\pi(3-t)^2$

$m_2(t) = \frac{-1}{10\pi} 10\pi(2)(3-t)$

$= -2(3-t)$

For $3 \le t \le 4$ $\quad m_2(t) = \frac{1}{10\pi}(10\pi) = 1$

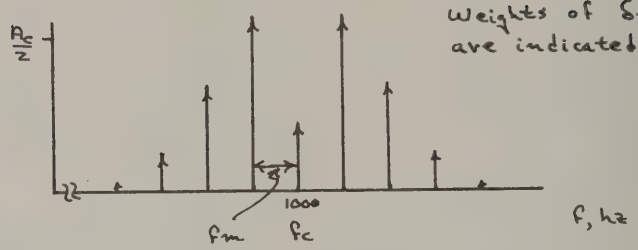


(3.29)

(a) $\beta = \frac{f_d A}{f_m} = \frac{5.4}{10} = 2.$

(b) $f_c = 1$ khz., $f_m = 10$ hz.

n	0	1	2	3	4
$J_n(2)$	.224	.577	.353	.129	.034



(c) This is not narrowband FM because components above $f_c + f_m$ and below $f_c - f_m$ contain significant power.

(d) $\beta = K_p \cdot A \Rightarrow 2 = K_p \cdot 4 \Rightarrow K_p = 1/2$

(3.30)

$$x_c(t) = A_c \cos\left[\omega_c t + \underbrace{K_f \int^t m(\alpha)\, d\alpha}_{\phi(t)}\right].$$

$m(t) = A\cos\omega_1 t + B\cos\omega_2 t$

$$\therefore \phi(t) = K_f \int^t [A\cos\omega_1 t + B\cos\omega_2 t]\, dt$$

$$= \frac{AK_f}{\omega_1}\sin\omega_1 t + \frac{BK_f}{\omega_2}\sin\omega_2 t.$$

Define $\beta_1 = \frac{AK_f}{\omega_1}$, $\beta_2 = \frac{AK_f}{\omega_2}$.

Then $x_c(t) = \mathrm{Re}\left\{A_c e^{j\omega_c t} e^{j\beta_1 \sin\omega_1 t} e^{j\beta_2 \sin\omega_2 t}\right\}$.

We can express $e^{j\beta_1\cos\omega_1 t}$, $e^{j\beta_2\cos\omega_2 t}$ as the Fourier series $\sum_{n=-\infty}^{\infty} J_n(\beta_1) e^{jn\omega_1 t}$, $\sum_{K=-\infty}^{\infty} J_K(\beta_2) e^{jK\omega_2 t}$.

Then $x_c(t)$ can be expressed as

(3.30) (Cont.)

$$x_c(t) = \mathrm{Re}\left\{ A_c e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta_1) e^{jn\omega_1 t} \sum_{K=-\infty}^{\infty} J_K(\beta_2) e^{jK\omega_2 t} \right\}$$

$$= \mathrm{Re}\left\{ A_c \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} J_n(\beta_1) J_K(\beta_2) e^{j(\omega_c + n\omega_1 + K\omega_2)t} \right\}$$

$$= A_c \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} J_n(\beta_1) J_K(\beta_2) \cos(\omega_c + n\omega_1 + K\omega_2)t .$$

(3.31)

$W = 15$ Khz, $\mathrm{Max}[m(t)] = 5$, $B = 2(D+1)W$ where

$D = f_d \, \mathrm{Max}[m(t)] / W = \dfrac{5 f_d}{15 \times 10^3}$.

	f_d	D	B
(a)	30	0.01	30.3 Khz
(b)	300	0.10	33 Khz
(c)	3000	1.0	60 Khz
(d)	30000	10.0	330 Khz

(3.32)

From Eqn. 3.83:

$$x_c(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

$$\therefore \quad \langle x_c^2(t) \rangle = \tfrac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta).$$

From Eqn. 3.89:

$$\langle x_c^2(t) \rangle = \tfrac{1}{2} A_c^2$$

$$\therefore \quad \tfrac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) = \tfrac{1}{2} A_c^2 \quad \text{and so}$$

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1 .$$

(3.33)

From (3.81)

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(nx-\beta\sin x)}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}\,dx$$

Thus

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\beta\sin x - nx)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(\beta\sin x - nx)\,dx$$

The imaginary part of $J_n(\beta)$ is zero since the integrand is an odd function of x and the limits are even $(-\pi, \pi)$. Thus

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\beta\sin x - nx)\,dx$$

Since the integrand is even

$$J_n(\beta) = \frac{1}{\pi}\int_{0}^{\pi}\cos(\beta\sin x - nx)\,dx$$

(3.34)

From the previous problem

$$J_n(\beta) = \frac{1}{\pi}\int_{0}^{\pi}\cos(\beta\sin x - nx)\,dx$$

with $\lambda = \pi - x$ we have

$$J_n(\beta) = \frac{1}{\pi}\int_{\pi}^{0}\cos\left[\beta\sin(\pi-\lambda) - n(\pi-\lambda)\right](-1)\,d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\cos\left[\beta\sin(\pi-\lambda) - n\pi + n\lambda\right]d\lambda$$

Since $\sin(\pi-\lambda) = -\sin(\lambda-\pi) = \sin\lambda$

(3.34) (cont.)

we have

$$J_n(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos\left[\beta \sin\lambda + n\lambda - n\pi\right] d\lambda$$

Since $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

with $u = \beta \sin\lambda + n\lambda$ and $v = n\pi$

$J_n(\beta)$ can be written

$$J_n(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos(\beta \sin\lambda + n\lambda)\cos(n\pi)\, d\lambda + \frac{1}{\pi}\int_0^{\pi} \sin(\beta \sin\lambda + n\lambda)\sin(n\pi)\, d\lambda$$

$\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$. Thus

$$J_n(\beta) = (-1)^n \frac{1}{\pi}\int_0^{\pi} \cos(\beta \sin\lambda + n\lambda)\, d\lambda$$

But $$J_{-n}(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos(\beta \sin\lambda + n\lambda)\, d\lambda$$

therefore

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

which is

$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

(3.35)

(a) Peak frequency deviation $= 8(10) = 80$

(b) $$\phi(t) = 2\pi(8)\int_0^t 10\cos(20\pi\alpha)\, d\alpha$$
$$= 16\pi(10)\frac{1}{20\pi}\sin 20\pi t$$
$$= 8\sin(20\pi t)$$

and the peak value of $\phi(t)$ is 8 rad.

(3.35) (Cont.)

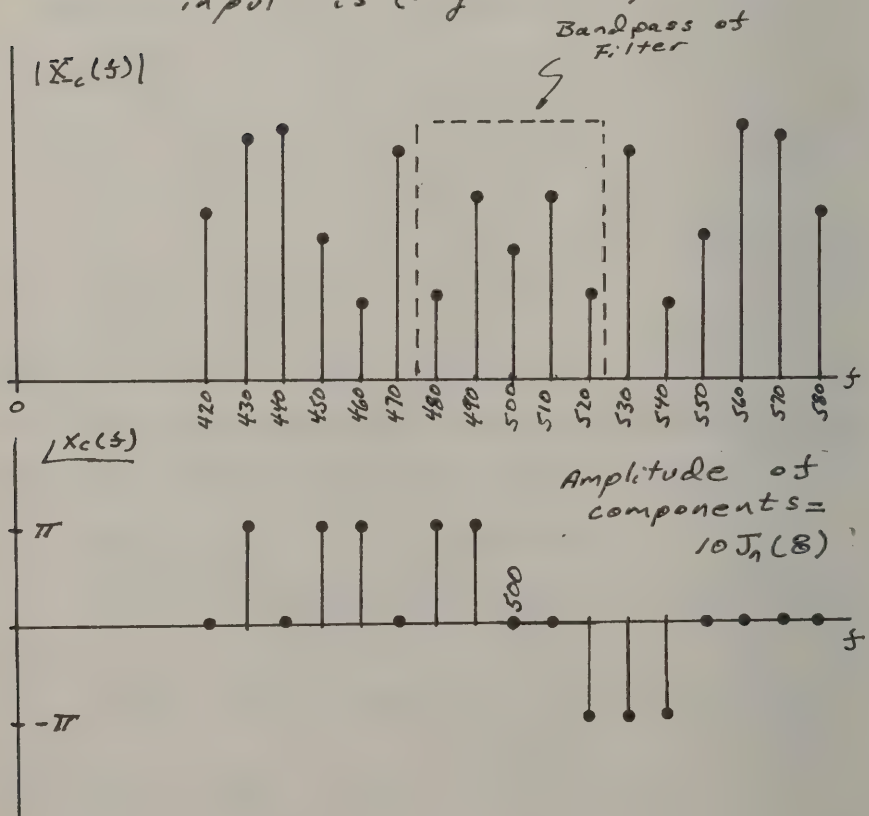
(c) The index β is the peak phase deviation. Thus $\beta = 8$ or

$$\beta = \frac{\text{Peak Frequency Dev.}}{f_m}$$

$$\beta = \frac{80}{10} = 8$$

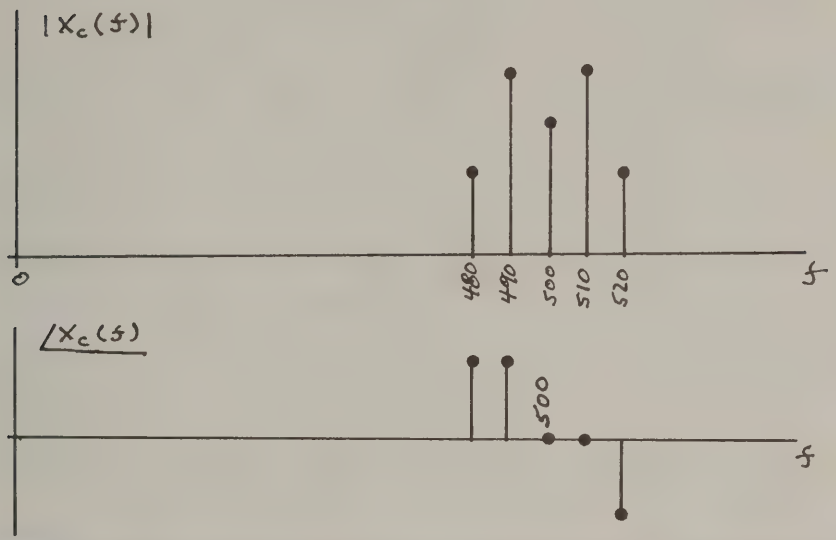
(d) The power at the filter input is $P_i = \frac{1}{2} A_c^2 = \frac{1}{2}(10)^2 = 50$ Watts

(e) The spectrum at the filter input is (single-sided)



(3.35) (cont.)

The single sided spectra at the filter output follows.



(2.36)

We need to calculate the power ratio P_r for the filter with index β. The index is

$\beta = \frac{700}{100} = 7$. Since $f_m = 100$ and the filter bandwidth is 1300 Hz, the first 6 components each side of the carrier are passed. Thus

$$P_r = J_0^2(7) + 2\sum_{n=1}^{6} J_n^2(7)$$

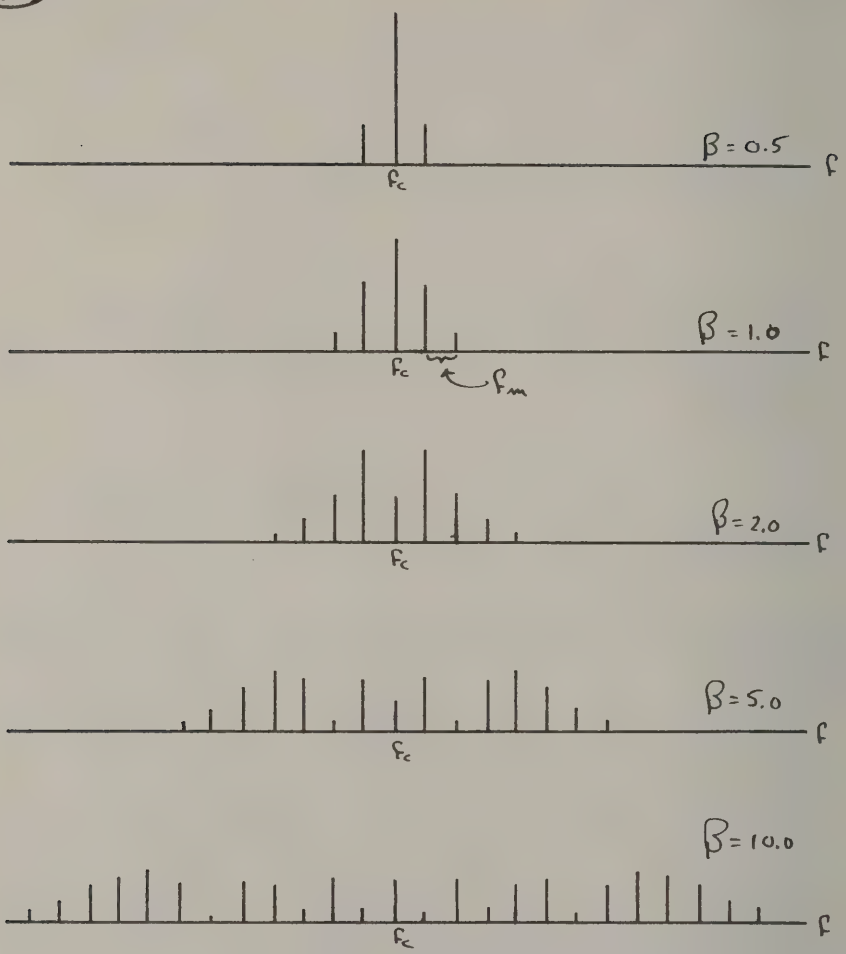
$J_n(7)$ is computed by

$$J_{n+1}(7) = \frac{2n}{7} J_n(7) - J_{n-1}(7)$$

The table is at right using the values in the table $P_r = 0.8496$ and $P_o = (2000)(0.8496) = 1699.2$ Watts

n	$J_n(7)$
0	0.3001
1	-0.0047
2	-0.3014
3	-0.1675
4	0.1578
5	0.3478
6	0.3391

(3.37)



(3.38)

$f_m = 150$ Hz., $\beta = 10$, $P_r = 0.5$

$$P_r = J_0^2(10) + 2\sum_{n=1}^{K} J_n^2(10)$$

$$= (.246)^2 + 2(.043)^2 + 2(.255)^2 + 2(.058)^2 + 2(.220)^2 + 2(.234)^2 + 2(.014)^2 + 2(.217)^2$$

$$= .5019$$

Thus, $K = 7$ and so bandwidth $= 2Kf_m = 2.1$ KHz.

(3.39)

First find N from the deviation requirement.

$D_{out} = N \cdot D_{in} \Rightarrow N = \frac{20}{0.1} = 200.$

Thus, $f_{c,out} = N \cdot f_{c,in} = 200(100\ \text{Khz}) = 20\ \text{Mhz}.$

The mixer produces sum and difference frequencies so $f_{LO} = 80\ \text{Mhz}$ or $120\ \text{Mhz}$.

The bandpass filter is centered at $f_c = 100\ \text{Mhz}$.

B = bandwidth $= 2(D+1)W = 210\ \text{Khz}$.

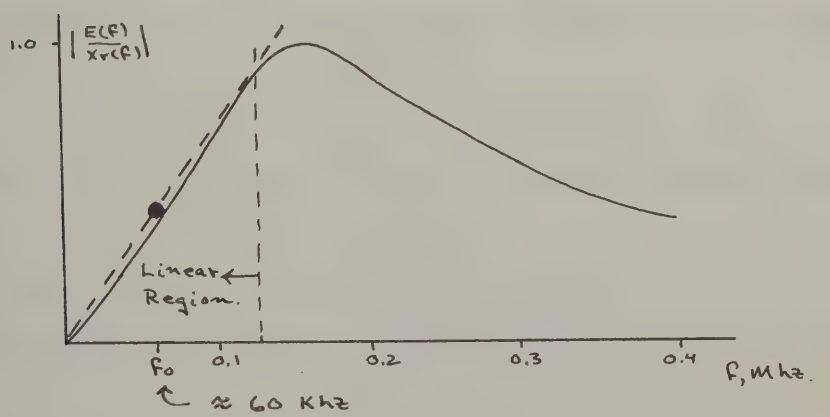
(3.40)

$$\frac{E(f)}{X_r(f)} = \frac{R}{R + j2\pi fL + \frac{1}{j2\pi fC}}.$$

$$\left|\frac{E(f)}{X_r(f)}\right| = \frac{1}{\left[1 + \left(2\pi f\tau_L - \frac{1}{2\pi f\tau_c}\right)^2\right]^{1/2}} \quad \text{where } \tau_L = \frac{L}{R},\ \tau_c = RC.$$

For $R = 10^3$, $L = 10^{-3}$, $C = 10^{-9}$: $\tau_L = 10^{-6}$ and $\tau_c = 10^{-6}$.

$$\therefore \left|\frac{E(f)}{X_r(f)}\right| = \frac{1}{\left[1 + \left(2\pi f - \frac{1}{2\pi f}\right)^2\right]^{1/2}} \quad \text{where } f \text{ is in Mhz.}$$



$K_D \approx 7.0 \times 10^{-6}$ v/hz. Max freq. deviation $\approx$ 60 Khz.

(3.41)

From Prob 3.40, the transfer function of the network is

$$\left|\frac{E(f)}{X_r(f)}\right| = \frac{1}{\left[1+\left(2\pi f\tau_L - \frac{1}{2\pi f\tau_c}\right)^2\right]^{1/2}} \quad \text{where } \tau_L = \frac{L}{R},\ \tau_c = RC.$$

Note that the resonant peak occurs at f_p where $2\pi f_p \tau_L - \frac{1}{2\pi f_p \tau_c} = 0 \Rightarrow f_p = \frac{1}{2\pi\sqrt{LC}}$.
Also, the smaller the value of R, the more peaked the response.
As a first attempt, let $f_p = 75$ Mhz. Then $LC = 4.5\times10^{-18}$.
Let $R = 10\,\Omega$, $L = 10^{-6}$ H, $C = 4.5\times10^{-12}$ f. Then
$\tau_L = 10^{-7}$, $\tau_c = 4.5\times10^{-11}$ and

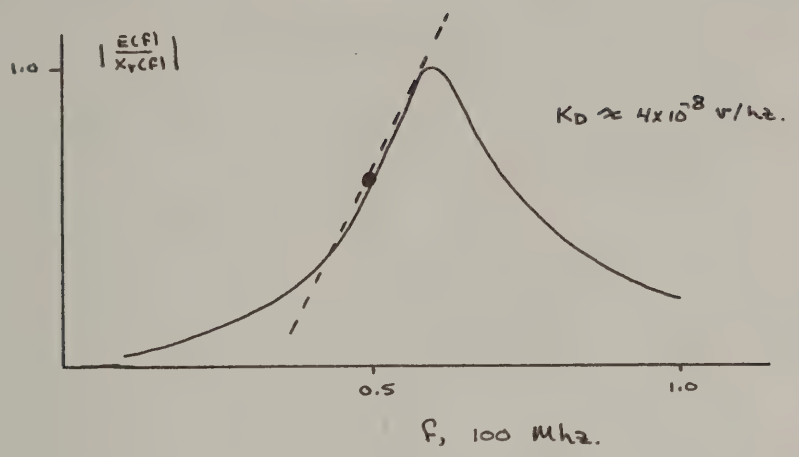
$$\left|\frac{E(f)}{X_r(f)}\right| = \frac{1}{\left[1+\left(20\pi f - \frac{10^3}{9\pi f}\right)^2\right]^{1/2}} \quad \text{where f is in 100 Mhz.}$$

Examination of this transfer function reveals that the linear region is much too far above 50 Mhz.
So let $f_p = 60$ Mhz. Then $LC = 7\times10^{-18}$.
Let $R = 10$, $L = 10^{-7}$, $C = 7\times10^{-11}$. Then $\tau_L = 10^{-8}$, $\tau_c = 7\times10^{-10}$.
Then

$$\left|\frac{E(f)}{X_r(f)}\right| = \frac{1}{\left[1+\left(2\pi f - \frac{14.3}{2\pi f}\right)^2\right]^{1/2}} \quad \text{where f is in 100 Mhz}$$

This function is plotted below.

(3.41) (Cont.)



Note that there is no unique solution to this problem. In fact, many values of τ_L, τ_c could be used and these time constants can be arrived at by many component values.

(3.42)

From Example 3.4

$$H(f) = \frac{j\,2\pi f RC}{1 + j\,2\pi f RC}$$

which, if $f \ll \frac{1}{2\pi RC}$, can be approximated

$$H(f) = j\,2\pi f RC.$$

Using the quasi-steady-state approximation

$$H\left(f_c + \tfrac{1}{2\pi}\tfrac{d\phi}{dt}\right) = j2\pi\left(f_c + \tfrac{1}{2\pi}\tfrac{d\phi}{dt}\right)RC$$

$$= j\left(\omega_c + \tfrac{d\phi}{dt}\right)RC$$

(3.42) (Cont.)

the envelope is

$$e(t) = \left(\omega_c + \frac{d\phi}{dt}\right) RC$$

and will be the approximate output of a linear envelope detector if $\frac{d\phi}{dt} < \omega_c$ for all t. Assuming ac coupling so that the dc component, $RC\omega_c$, is removed

$$y_D(t) = \frac{d\phi}{dt} RC = 2\pi f_d RC\, m(t).$$

Thus, we have an FM discriminator with $K_D = 2\pi f_d RC$. This is the same result obtained in Example 3.4

(3.43)

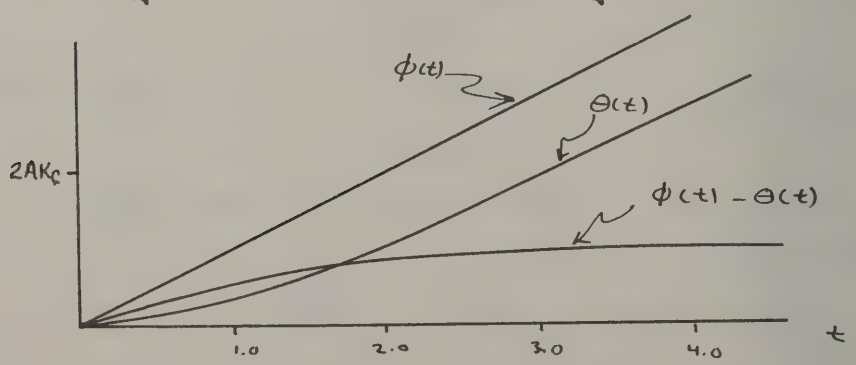
(a) $\phi(t) = AK_f \int_{-\infty}^{t} u(\alpha)\, d\alpha = A K_f t.$

$\Theta(s) = \frac{A K_t K_f}{s^2(s+K_t)}$. Laplace transforming gives

$\Theta(t) = A K_t K_f \frac{1}{K_t^2}\left(K_t t - 1 + e^{-K_t t}\right) u(t) = AK_f\left(t + \frac{1}{K_t} e^{-K_t t} - \frac{1}{K_t}\right) u(t).$

The phase error $= \phi(t) - \Theta(t) = AK_f\left[\frac{1}{K_t}\left(1 - e^{-K_t t}\right)\right] u(t).$

Letting $K_t = 1$, the following plot is obtained.



(3.43) (Cont.)

(b) From part (a), the phase error is

$$\phi(t) - \theta(t) = A\frac{K_f}{K_t}\left[1 - e^{-K_t t}\right].$$

The maximum phase error occurs for $t \to \infty$ and is AK_f/K_t. Thus,

$$0.1 \text{ rad.} \rightarrow K_t = 10\,A\,K_f.$$

The VCO gain K_v is contained in K_t.

(3.44)

With $m(t) = A\cos\omega_m t$ we have

$$\phi(t) = AK_f\int^t \cos\omega_m t\,dt = \frac{AK_f}{\omega_m}\sin\omega_m t$$

and

$$\Phi(s) = \frac{AK_f}{\omega_m}\,\frac{\omega_m}{s^2+\omega_m^2}.$$

The phase of the VCO output is

$$\theta_s = \Phi(s)\frac{K_t}{s+K_t} = AK_fK_t\,\frac{1}{(s+K_t)(s^2+\omega_m^2)}.$$

Using partial fraction expansion

$$\frac{1}{(s+K_t)(s^2+\omega_m^2)} = \frac{K_1}{s+K_t} + \frac{K_2 s + K_3}{s^2+\omega_m^2}$$

where

$$K_1 = \frac{1}{K_t^2+\omega_m^2}$$

$$1 = \frac{s^2+\omega_m^2}{K_t^2+\omega_m^2} + (K_2 s + K_3)(s+K_t)$$

$$1 = \frac{s^2+\omega_m^2}{K_t^2+\omega_m^2} + K_2 s^2 + (K_2K_t + K_3)s + K_3K_t$$

so that

$$K_2 = \frac{-1}{K_t^2+\omega_m^2}, \quad K_3 = \frac{K_t}{K_t^2+\omega_m^2}$$

(3.44) (Cont.)

Thus

$$\Theta(s) = \frac{A K_f K_t}{K_t^2 + \omega_m^2}\left[\frac{1}{s+K_t} - \frac{s}{s^2+\omega_m^2} + \frac{K_t}{s^2+\omega_m^2}\right]$$

so that

$$\theta(t) = \frac{A K_f K_t}{K_t^2 + \omega_m^2}\left[e^{-K_t t} - \cos\omega_m t + \frac{K_t}{\omega_m}\sin\omega_m t\right] u(t)$$

From (3.123) the VCO input signal is

$$e_v(t) = \frac{1}{K_v}\frac{d\theta}{dt}$$

which is

$$e_v(t) = \frac{A K_f K_t}{K_v(K_t^2+\omega_m^2)}\left[-K_t e^{-K_t t} + \omega_m \sin\omega_m t + K_t \cos\omega_m t\right] u(t)$$

The first term in this expression is the transient response. If K_t is large, only the third term is significant. Also, for large K_t, $K_t^2 + \omega_m^2 \approx K_t^2$, so that the steady-state response is

$$e_v(t) \approx \frac{A K_f K_t}{K_t^2 K_v}\left[K_t \cos\omega_m t\right] u(t)$$

$$= \frac{K_f}{K_v} A \cos\omega_m t \qquad t>0$$

which is proportional to $m(t)$ and is the desired demodulated output. Note that if the transmitter and VCO deviation constants are equal then

$$e_v(t) \approx m(t).$$

(3.45)

The Costas PLL is illustrated in Fig. 3.47. The output of the top multiplier is

$$m(t)\cos\omega_c t\,[2\cos(\omega_c t+\theta)] = m(t)\cos\theta + m(t)\cos(2\omega_c t+\theta)$$

so that the output of the lowpass filter is $m(t)\cos\theta$. The output of the quadrature multiplier is

$$m(t)\cos\omega_c t\,[2\sin(\omega_c t+\theta)] = m(t)\sin\theta + m(t)\sin(2\omega_c t+\theta)$$

so that the output of the lowpass filter is $m(t)\sin\theta$. The multiplication

$$m^2(t)\cos\theta\sin\theta$$

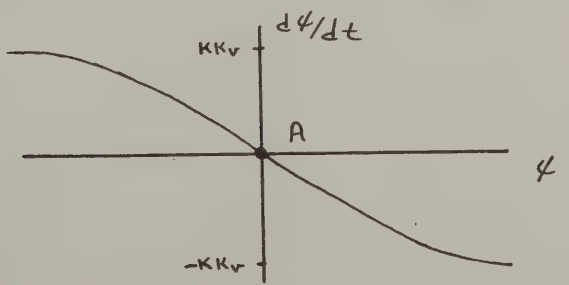
yields $m^2(t)\sin 2\theta$ as indicated. Note that with assumed input $m(t)\cos\omega_c t$ and assumed VCO output $2\cos(\omega_c t+\theta)$, the phase error is $\theta(t)$. Thus, the VCO is defined by

$$\frac{d\theta}{dt} = \frac{d\psi}{dt} = -K_v e_r(t).$$

Since $e_r(t) = -K\sin 2\theta$, the phase plane is defined by

$$\frac{d\psi}{dt} = -KK_v\sin 2\psi$$

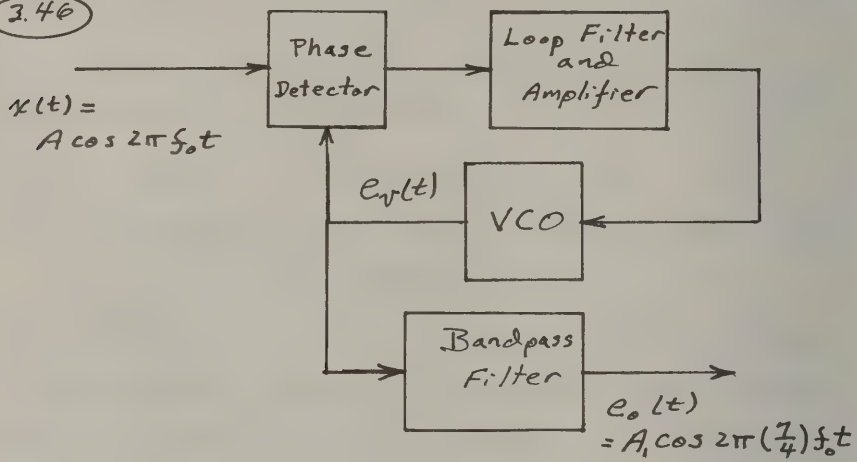
and is as shown.



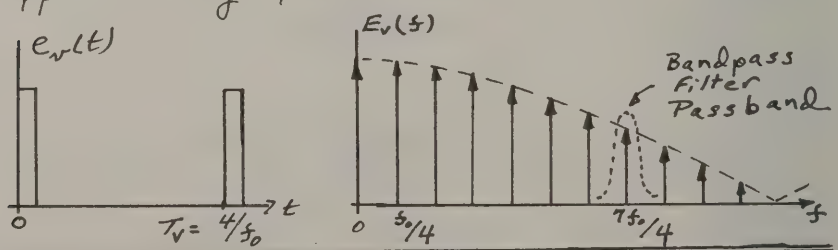
3.45 (cont.)

Since the $\frac{d\psi}{dt}$ intersection is on a portion with negative slope, the point A is a stable operating point. Thus the loop locks with zero phase and frequency error.

3.46

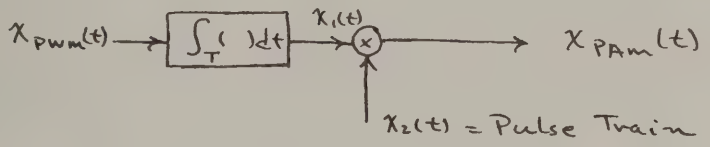


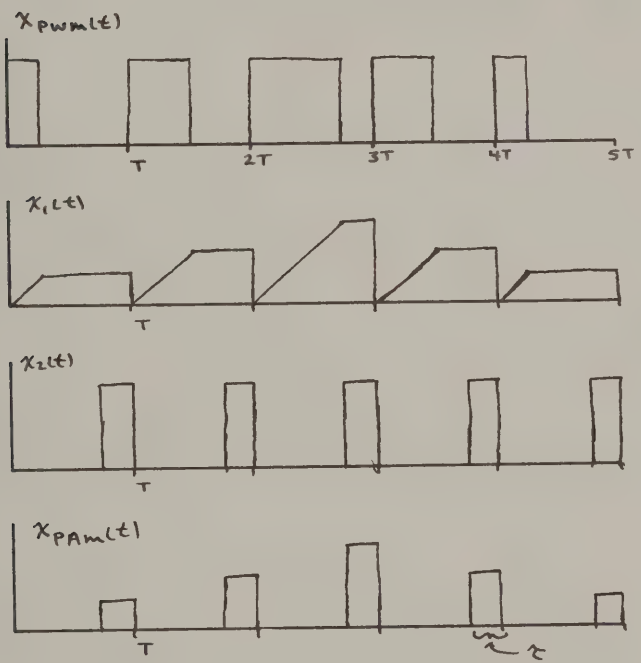
The fundamental of the VCO output has frequency $f_v = \frac{1}{T_v} = \frac{1}{4} f_o$. The spectrum of the VCO output consists of components separated by $\frac{1}{4} f_o$ with envelope sinc $x f_o$. The center frequency of the filter is $\frac{7}{4} f_o$ and the bandwidth of the filter is approximately $\frac{1}{4} f_o$ as shown below.



(3.47)

Conceptually, a system for converting PWM to PAM can be realized as shown.



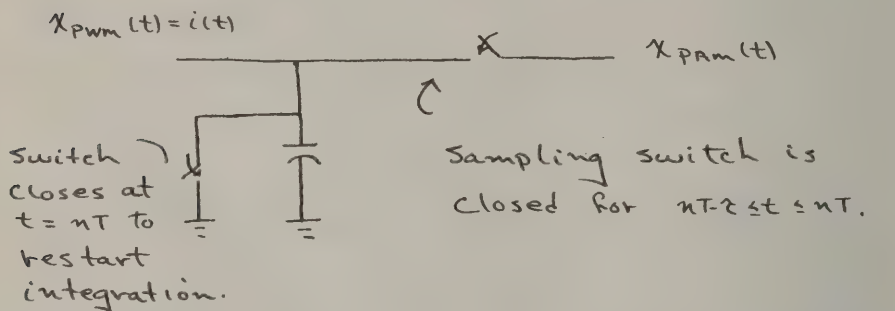


(3.47) (Cont.)

The integrator can be realized by using a capacitor since, for a capacitor,

$$v(t) = \frac{1}{C}\int i(t)\,dt$$

Multiplication by the pulse train can be realized by sampling the capacitor voltage. Thus, the simple circuit is



(3.48)

Let the peak-to-peak value of the data signal be A. Since the maximum peak-to-peak error is 2% of A, 0.02 A, the maximum width of a quantization level is 0.02A. Thus the required number of quantization levels is

$$\frac{A}{0.02A} = 50 \leq 2^N.$$

Thus we let $N=6$, which gives $q=64$ quantization levels.

The bandwidth is estimated using (3.169)

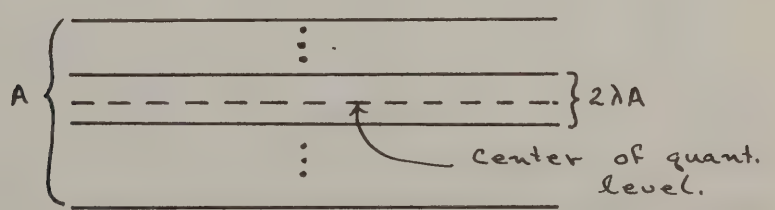
$$B \approx 2W \log_2 q = 2(4\times10^3)6 = 48\text{KHz}.$$

Note: See next problem.

(3.49)

Let the maximum error be λA, where A is the peak-to-peak value of the message signal. Thus, the allowable peak-to-peak error is $2\lambda A$, and the minimum number of quantization levels is

$$q_{min} = \frac{A}{2\lambda A} = \frac{1}{2\lambda}.$$



(3.49) (Cont.)

The maximum error is λA assuming that a quantization level is decoded by assuming that the sample lies in the center of a level.

The wordlength N is given by

$$[N] = \log_2 \frac{1}{2\lambda}$$

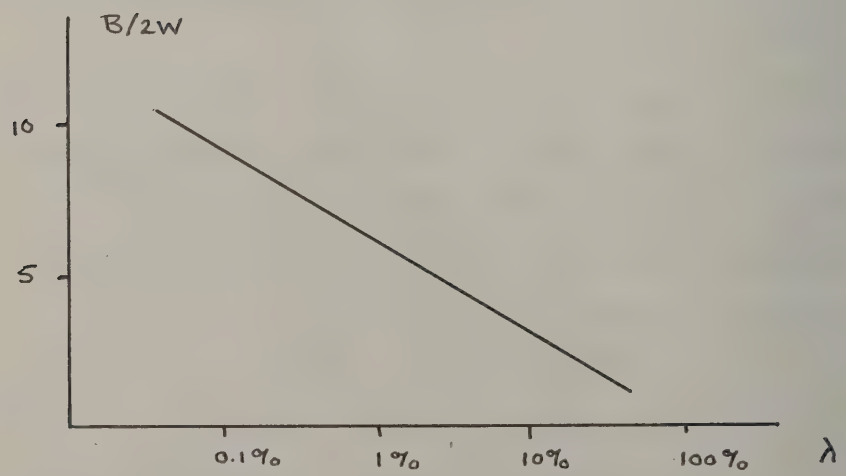
where $[N]$ is the smallest integer greater than or equal to $\log_2 \frac{1}{2\lambda}$. The bandwidth is, from (3.169)

$$B \approx 2W[N] = 2W\left[\log_2 \frac{1}{2\lambda}\right]$$

or

$$\frac{B}{2W} = \left[\log_2 \frac{1}{2W}\right].$$

This is plotted below for $0.001 \leq \lambda \leq 0.5$.



Note: For $\lambda = 1\%$, $B/2W = 6$. This is the previous problem.

(3.50)

Type	Advantages	Disadvantages
PAM	Very easy to implement the modulator (sampler) and demodulator (low pass filter). Synchronization is not required.	We will show in Chap. 6 that PAM has poor noise performance at high signal-to-noise ratio
PWM	Simple demodulators (lowpass filters). Synchronization is not required. We will see in Chap. 6 that PWM has good noise performance	The modulato more compl for PAM.
PPM	Since the pulses can be made narrow, the power in X can be made We shall s Chap. 6 th good noi	
PCM	Exce fo	

(3.51) $m(t) = 3 \sin 2\pi(10)t + 4 \sin 2\pi(20)t$

$$\frac{dm}{dt} = 3(20\pi)\cos 20\pi t + 4(40\pi)\cos 40\pi t$$

$$\frac{\delta_0}{2\pi} = \delta_0 f_s = 0.1\pi f_s \geq \text{Max}\left|\frac{dm}{dt}\right| = 60\pi + 160\pi$$

$$0.1\pi f_s \geq 220\pi$$

$$\therefore \frac{220\pi}{0.1\pi} = 2200$$

$+ f_s$ to

Hz

The need for synchronization results in more complicated demodulators.

Small. ee in t PPM has se performance

lent noise Performance in high noise environment. See (3.47)

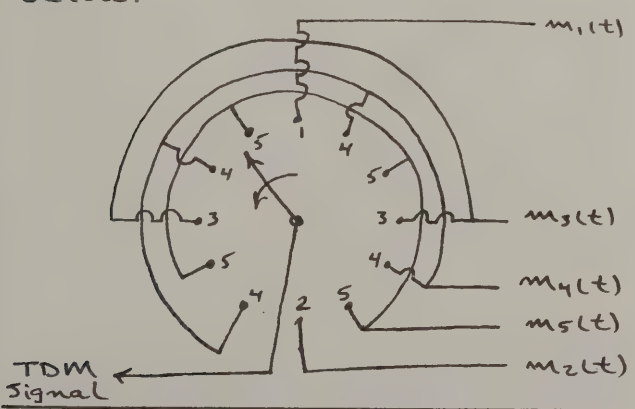
Synchronization required. The A/D converters increase complexity

(3.53)

Assume that the sampling commutator rotates at a frequency of 2W Hz. With Nyquist rate sampling we get the following table.

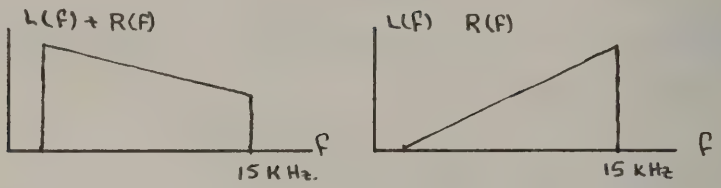
Signal $m_i(t)$	Bandwidth of $m_i(t)$	Samples per rev.
$m_1(t)$	W	1
$m_2(t)$	W	1
$m_3(t)$	2W	2
$m_4(t)$	4W	4
$m_5(t)$	4W	4

Note that the table indicates that the bandwidth of TDM signal is 12W and the number of commutator poles must be at least 12.
The commutator hookup is shown below.

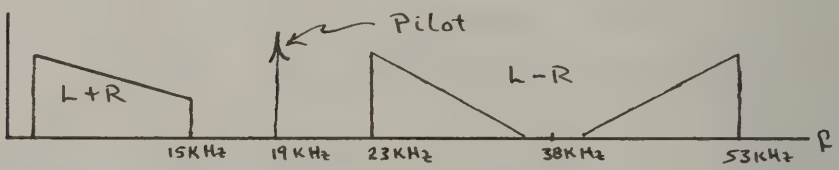


(3.54)

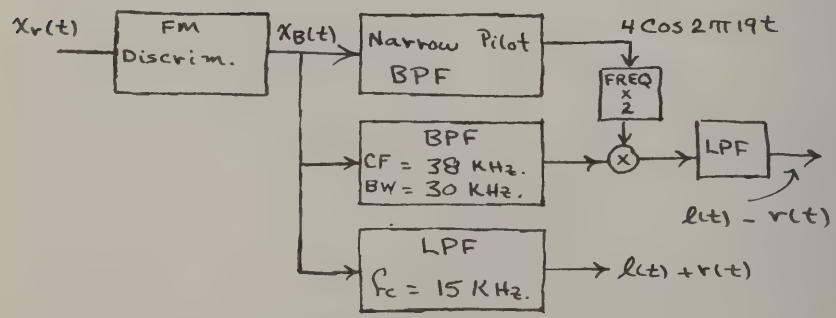
(a) The modulation is assumed to have the spectrum

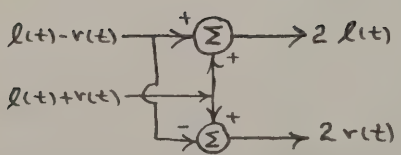


We've assumed the phase of L(f), R(f) zero so that we can add the spectra. The baseband spectrum is:



(b) The receiver can be implemented as shown below.





(3.54) (Cont.)

(c) For conventional FM, simply lowpass filter $x_B(t)$ to obtain $l(t) + r(t)$.

(d) By Carson's rule $B = 2(D+1)W$.

Thus

For stereo: $W = 53\,KHz \rightarrow B = 636\,KHz$.

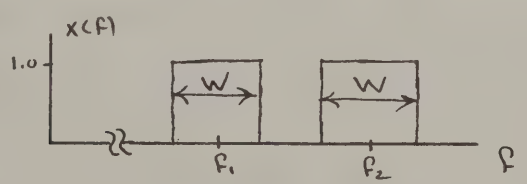
For non-stereo: $W = 15\,KHz \rightarrow B = 180\,KHz$.

Note: It is clear from this problem that the deviation ratio should, for most applications be reduced when utilizing stereophonic transmission.

For commercial FM broadcasting the allowable peak frequency deviation is 75KHz for either nonstereophonic or stereophonic applications. Thus for nonstereophonic applications $D = 75\,KHz/15\,KHz = 5$, while for stereophonic applications we have $D = 75\,KHz./53\,KHz = 1.4$.

(3.55)

The single-sided spectrum for $x(t)$ is



Since

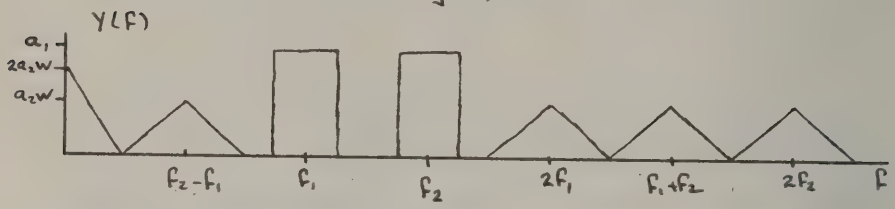
$$y(t) = a_1 x(t) + a_2 x^2(t)$$

(3.55) (cont.)

and

$$Y(f) = a_1 X(f) + a_2 X(f) * X(f)$$

the spectrum of y(t) is



Demodulation can be a problem since it may be difficult to filter the desired signals from the signals resulting from the nonlinearity. As more signals are included in x(t) the problem becomes worse.

The difficulty with harmonically related carriers is that portions of the spectrum of Y(f) overlap. This can be seen by examining the spectrum of Y(f) drawn above when $f_2 = 2f_1$. For this case the distortion spectrum centered at $f_2 - f_1$ now falls exactly on top of the desired spectrum centered at f_1.

(4.1)

Consider events A, B mutually exclusive and from sample space S.

(1) $P(A) \triangleq \lim_{N\to\infty} \frac{N_A}{N}$, $N_A \geq 0, N > 0$ so $\frac{N_A}{N} \geq 0$ and $P(A) \geq 0$.

(2) $P(S) \triangleq \lim_{N\to\infty} \frac{N}{N} = 1$.

(3) $P(A+B) = \lim_{N\to\infty} \frac{N_{A\bar{B}} + N_{\bar{A}B} + N_{AB}}{N}$. But $N_{A\bar{B}} = N_A$, $N_{\bar{A}B} = N_B$, $N_{AB} = 0$ so $P(A+B) = \lim_{N\to\infty} \frac{N_A + N_B}{N} = \lim_{N\to\infty} \frac{N_A}{N} + \lim_{N\to\infty} \frac{N_B}{N}$

$= P(A) + P(B)$.

(4.2)

Possible outcomes are: (1) Dime chosen, (2) Nickel chosen.

Let A = selection of nickel, B = selection of dime.

Then S = sample space composed of all combinations of coins and A, B are exhaustive and mutually exclusive.

$P(A) = \frac{25}{40}$, $P(B) = \frac{15}{40}$.

(4.3)

S = sample space is the collection of the 20 parts.

Let A_i = the event that pointer stops on i^{th} part.

Then A_i are exhaustive and mutually exclusive.

$P(A_i) = \frac{1}{20}$.

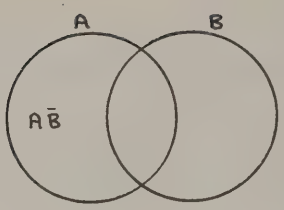
(a) $P(\text{odd no.}) = P(A_1 + A_3 + \ldots + A_{19}) = \frac{10}{20}$

(b) $P(A_9) = \frac{1}{20}$.

(c) $P(A_5 + A_9) = \frac{1}{20} + \frac{1}{20} = \frac{2}{20}$

(d) $P(\text{no. less than } 12) = P(A_1 + A_2 + \ldots A_{11}) = \frac{11}{20}$.

(4.4)



(a) $A+B = (A\bar{B})+B$:

Also, $A\bar{B}$ and B are mutually exclusive.

So $P(A+B) = P(A\bar{B}+B) = P(A\bar{B}) + P(B)$

$= P(A\bar{B}) + P(B) + P(AB) - P(AB) = \underbrace{P(A\bar{B}) + P(AB)}_{\text{mutually exclusive}} + P(B) - P(AB)$

$= P(A\bar{B} + AB) + P(B) - P(AB) = P[\underbrace{A(\bar{B}+B)}_{S}] + P(B) - P(AB)$

$= P(A) + P(B) - P(AB)$

(b) By multiple application of part (a):

$P(A+B+C) = P[A+(B+C)] = P(A) + P(B+C) - P[A(B+C)]$

$= P(A) + P(B) + P(C) - P(BC) - P(AB+AC)$

$= P(A) + P(B) + P(C) - P(BC) - \{P(AB) + P(AC) - P[(AB)(BC)]\}$

$= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) + P(ABC).$

(4.5)

(a) Let A = event a King on first draw

B = event a King on second draw.

Then $P(AB) = P(B/A)\,P(A) = \frac{3}{51}\cdot\frac{4}{52} = \frac{12}{2652} = \frac{1}{221}$

(b) Let A_i = event an ace on i^{th} draw.

B_i = event a jack on i^{th} draw.

Then $P(A_1B_2 + B_1A_2) = P(A_1B_2) + P(B_1A_2)$

$= P(B_2/A_1)\,P(A_1) + P(A_2/B_1)\,P(B_1) = \frac{4}{51}\cdot\frac{4}{52} + \frac{4}{51}\cdot\frac{4}{52} = \frac{32}{2652} = \frac{8}{663}$

(c) Let A_i = event a heart on i^{th} draw.

$P(A_1A_2) = P(A_2/A_1)\,P(A_1) = \frac{12}{51}\cdot\frac{13}{52} = \frac{156}{2652} = \frac{1}{17}$

(4.5) (Cont.)

(d) Let A_i = event an ace on i^{th} draw,

B_i = event a club on i^{th} draw.

Then $P(A_1B_2 + B_1A_2) = P(A_1B_2) + P(B_1A_2) - P(A_1B_2B_1A_2)$ (0)

$= P[A_1B_2(B_1+\bar{B}_1)] + P[B_1A_2(A_1+\bar{A}_1)]$

$= P(A_1B_1B_2 + A_1\bar{B}_1B_2) + P(A_1B_1A_2 + \bar{A}_1B_1A_2)$

$= P(A_1B_1B_2) + P(A_1\bar{B}_1B_2) + P(A_1B_1A_2) + P(\bar{A}_1B_1A_2)$

$- P(A_1B_1B_2A_1\bar{B}_1B_2) - P(A_1B_1A_2\bar{A}_1B_1A_2)$ (0) (0)

$= P(B_2/A_1B_1)\,P(A_1B_1) + P(B_2/A_1\bar{B}_1)\,P(A_1\bar{B}_1)$

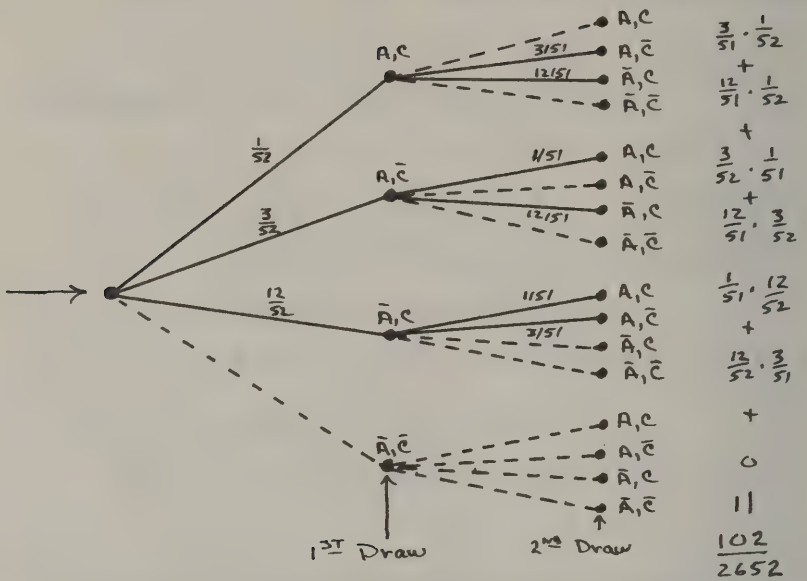
$+ P(A_2/A_1B_1)\,P(A_1B_1) + P(A_2/\bar{A}_1B_1)\,P(\bar{A}_1B_1)$

$= \frac{12}{51}\cdot\frac{1}{52} + \frac{13}{51}\cdot\frac{3}{52} + \frac{3}{51}\cdot\frac{1}{52} + \frac{4}{51}\cdot\frac{12}{52} = \frac{102}{2652} = \frac{1}{26}$

An alternate (and conceptually easier) approach to this type problem is the use of the tree diagram. The diagram is traversed from left to right. Nodes denote outcomes and the probability of traversing a branch between two nodes is listed along side the branch. Branches that lead to favorable outcomes are solid whereas the others are broken lines. The total probability of success is computed by multiplying the probabilities along each path, then adding these together. The tree diagram for this problem is shown below where the

(4.5) (Cont.)

Symbol A denotes Ace and C denotes Club.



(4.6)

$P(A/B) = P(A)$.

Multiplying by $P(B)$ gives $P(A/B)P(B) = P(A)P(B)$

By Bayes' Thm $P(A/B)P(B) = P(AB) = P(B/A)P(A)$. Thus

$P(B/A)P(A) = P(A)P(B) \Rightarrow P(B/A) = P(B)$.

(4.7)

$P(A_1A_2) = P(A_1)P(A_2)$, $P(A_1A_3) = P(A_1)P(A_3)$

$P(A_1A_4) = P(A_1)P(A_4)$, $P(A_2A_3) = P(A_2)P(A_3)$

$P(A_2A_4) = P(A_2)P(A_4)$, $P(A_3A_4) = P(A_3)P(A_4)$

$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3)$, $P(A_2A_3A_4) = P(A_2)P(A_3)P(A_4)$

$P(A_1A_3A_4) = P(A_1)P(A_3)P(A_4)$, $P(A_1A_2A_4) = P(A_1)P(A_2)P(A_4)$

$P(A_1A_2A_3A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$.

(4.8)

(a) Note that $A_1 = A_1(A_2+\bar{A}_2) = A_1A_2 + A_1\bar{A}_2$ and A_1A_2, $A_1\bar{A}_2$ are mutually exclusive.

Thus $P(A_1) = P[A_1(A_2+\bar{A}_2)] = P(A_1A_2+A_1\bar{A}_2) = P(A_1A_2) + P(A_1\bar{A}_2)$

$\therefore P(A_1\bar{A}_2) = P(A_1) - P(A_1A_2) = P(A_1)[1-P(A_2)] = P(A_1)P(\bar{A}_2)$

(b) $P(\bar{A}_1) = P[\bar{A}_1(A_2+\bar{A}_2)] = P(\bar{A}_1A_2) + P(\bar{A}_1\bar{A}_2)$

$\therefore P(\bar{A}_1\bar{A}_2) = P(\bar{A}_1) - P(\bar{A}_1)P(A_2) = P(\bar{A}_1)[1-P(A_2)] = P(\bar{A}_1)P(\bar{A}_2)$.

(4.9)

A, B mutually exclusive $\Rightarrow P(A/B) = P(B/A) = 0$.

A, B statistically independent $\Rightarrow P(B/A) = P(B), P(A/B) = P(A)$.

$\therefore P(A) = P(B) = 0$.

(4.10)

Let first event be A, second event be B.

Examine $P(B/A)$. $P(B/A) = P(B) \Rightarrow$ independence.

(a) $P(B/A) = 1 \neq P(B) = \frac{26}{52}$, Not ind.

(b) $P(B/A) = 0 \neq P(B) = \frac{26}{52}$. Not ind.

(c) $P(B/A) = \frac{2}{4} = P(B) = \frac{26}{52}$. Ind.

(d) $P(B/A) = 1 \neq P(B) = \frac{4}{13}$. Not ind.

(4.11)

$P(\bar{A}) = 1-P(A) = 0.5$, $P(\bar{B}/A) = 1-P(B/A) = 0.1$, $P(B/\bar{A}) = 1-P(\bar{B}/\bar{A}) = 0.1$

$P(B) = P(AB) + P(\bar{A}B) = P(B/A)P(A) + P(B/\bar{A})P(\bar{A}) = 0.5$

$P(\bar{B}) = 1-P(B) = 0.5$

$P(A/B) = \frac{P(B/A)P(A)}{P(B)} = 0.9$, $P(A/\bar{B}) = \frac{P(\bar{B}/A)P(A)}{P(\bar{B})} = 0.1$

(4.12)

Let A be the event student passes quiz.

B be the event student studied.

$P(A/B) = 0.9$, $P(A/\bar{B}) = 0.2$, $P(B) = 0.6$

$P(\bar{B}) = 1 - P(B) = 0.4$

$P(A) = P(A\bar{B}) + P(AB) = P(A/\bar{B})\,P(\bar{B}) + P(A/B)\,P(B) = 0.62$

$P(B/A) = \dfrac{P(A/B)\,P(B)}{P(A)} = 0.87$

(4.13)

(a) Computations in the order entered into table:

$\Sigma P(A_i) = 1 \Rightarrow P(A_1) = 0.2$

$P(A_1) = \Sigma P(A_1, B_i) \Rightarrow P(A_1, B_1) = 0.05$

$P(B_1) = \Sigma P(A_i, B_1) \Rightarrow P(A_3, B_1) = 0$

$P(A_3) = \Sigma P(A_3, \bar{B}_i) \Rightarrow P(A_3, \bar{B}_3) = 0.2$

$P(B_3) = \Sigma P(A_i, \bar{B}_3) \Rightarrow P(A_2, \bar{B}_3) = 0.05$

$P(A_2) = \Sigma P(A_2, \bar{B}_i) \Rightarrow P(A_2, B_2) = 0.4$

$P(B_2) = \Sigma P(A_i, \bar{B}_2) \Rightarrow P(B_2) = 0.6$

(b) $P(B_1/A_2) = \dfrac{P(A_2, B_1)}{P(A_2)} = 0.1$

$P(A_1/\bar{B}_3) = \dfrac{P(A_1, \bar{B}_3)}{P(\bar{B}_3)} = 0.17$

$P(B_1/A_1) = \dfrac{P(A_1, B_1)}{P(A_1)} = 0.25$

(4.14)

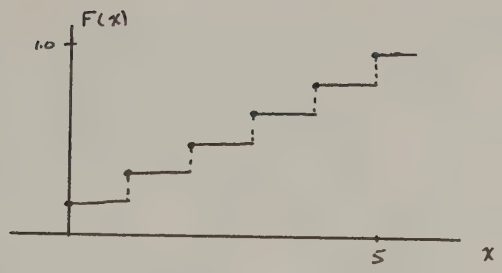
(a)

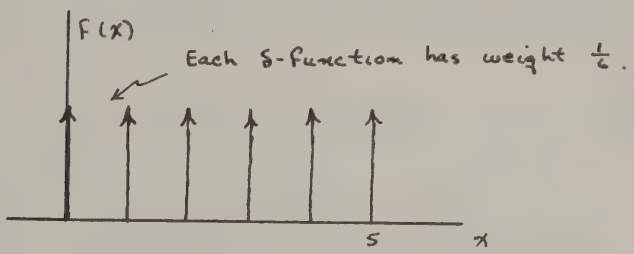
Die #2 ↓ / X_1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b)

Die #2 ↓ / X_2	1	2	3	4	5	6
1	1	0	1	0	1	0
2	0	1	0	1	0	1
3	1	0	1	0	1	0
4	0	1	0	1	0	1
5	1	0	1	0	1	0
6	0	1	0	1	0	1

4.15

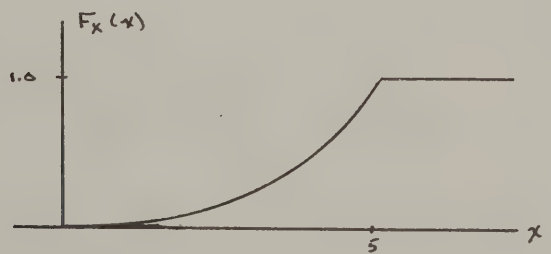


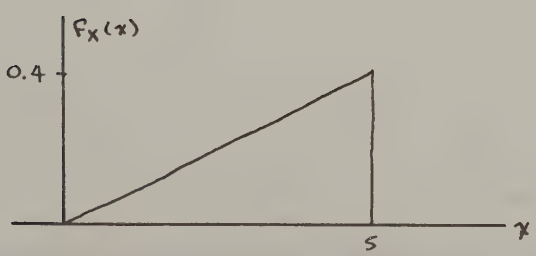


4.16

$F_X(+\infty) = 25A = 1$

$\therefore \quad A = 0.04$





4.17

(a) $f(x) \geq 0 \ \forall x$ and $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow f(x)$ is a p.d.f.

(b) $f(x) \geq 0 \ \forall x$ but $\int_{-\infty}^{\infty} f(x)dx = \infty \Rightarrow f(x)$ is not a p.d.f.

Limit x to $x \geq 0$, $f(x) = 0$ otherwise. Then $\int_{-\infty}^{\infty} f(x)dx = \frac{c}{\alpha}$.

If $c = \alpha$, then $f(x)$ is a p.d.f.

(c) $\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\pi} \tan^{-1} x \Big|_{-\infty}^{\infty} = 1, \Rightarrow f(x)$ is a p.d.f.

4.18

(a) Is a c.d.f.

(b) Is not a c.d.f. because is not continuous from right.

4.19

(a) By inspection $f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$ because $f_{XY}(x,y)$ is factorable.

(b) $f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$

$F_Y(y) = \int_{-\infty}^{y} f_Y(\alpha)d\alpha = \int_0^y e^{-\alpha}d\alpha = \begin{cases} 1 - e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$.

4.20

(a) X, Y are independent because $f_{XY}(x,y)$ factors to give

$f_X(x) = \begin{cases} a_1 e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $f_Y(y) = \begin{cases} a_2 e^{-y}, & y \geq 0 \\ 0, & y < 0, \end{cases}$

where $a_1 a_2 = \alpha$. Also, $a_1 = 1$, $a_2 = 1 \Rightarrow \alpha = 1$.

(b) X, Y are not independent because limits on x are a function of y so that $f_{XY}(x,y)$ is not factorable into terms depending solely on x and solely on y.

(4.21)

(a) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \int_0^4\int_0^3 C(1+xy)\,dx\,dy = 48C \Rightarrow C = \frac{1}{48}$

(b) $F_{XY}(0.5, 2.0) = \int_0^{2.0}\int_0^{0.5}\frac{1}{48}(1+xy)\,dx\,dy = \int_0^{2.0}\frac{1}{48}(.5+.125y)\,dy$

$= \frac{1}{48}\left(.5y + \frac{.125}{2}y^2\right)\Big|_0^{2.0} = .02604.$

(c) $f_{X,Y}(x,2) = f_{XY}(x,y)\big|_{y=2} = \frac{1}{48}(1+2x) \quad 0 \le x \le 3.$

(d) $f_{X/Y}(x/y) = \dfrac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx} = \dfrac{\frac{1}{48}(1+xy)}{\frac{1}{48}(3+\frac{9}{2}y)} = \dfrac{1+xy}{3+9/2\,y}$

$\therefore\ f_{X/Y}(x/y=2.0) = \frac{1+2x}{12}.$

(4.22)

(a) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} x e^{-x} e^{-xy}\,dy$

$= -e^{-x}e^{-xy}\Big|_{y=0}^{\infty} = \begin{cases} e^{-x} & 0 \le x \\ 0, & x < 0. \end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} x e^{-(y+1)x}\,dx$

$= \frac{1}{(y+1)^2}\left[-x(y+1)-1\right]\Big|_{x=0}^{\infty} = \begin{cases} \frac{1}{(y+1)^2}, & y \ge 0 \\ 0, & y < 0 \end{cases}.$

(b) The variables are dependent because $f_{XY}(x,y) \neq f(x)f(y)$.

(4.23)

$V(t) = X\cos\omega t + Y\sin\omega t = X\cos\omega t + Y\cos(\omega t + \pi/2)$

$= \mathrm{Re}\left\{X e^{j\omega t} + Y e^{j(\omega t+\pi/2)}\right\} = \mathrm{Re}\left\{(X+jY)e^{j\omega t}\right\}$

$= \mathrm{Re}\left\{\sqrt{X^2+Y^2}\,e^{j\theta}e^{j\omega t}\right\}$, where $\theta = \tan^{-1}(Y/X)$

$= R\cos(\omega t+\theta)$, where $R = \sqrt{X^2+Y^2}$.

$R = g_1(x,y) = \sqrt{x^2+y^2}$, $\theta = g_2(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$X = g_1^{-1}(R,\theta) = R\cos\theta$, $Y = g_2^{-1}(R,\theta) = R\sin\theta$.

(4.23) (Cont.)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R.$$

$$\therefore f_{R\Theta}(r,\theta) = f_{XY}(x,y) \frac{\partial(x,y)}{\partial(u,v)} \Big|_{\substack{x = g_1^{-1}(R,\theta) \\ y = g_2^{-1}(R,\theta)}}$$

$$= \frac{1}{2\pi\sigma^2} R e^{-\frac{1}{2\sigma^2}R^2\cos^2\theta} e^{-\frac{1}{2\sigma^2}R^2\sin^2\theta} = \frac{1}{2\pi\sigma^2} R e^{-\frac{1}{2}\left(\frac{R}{\sigma}\right)^2}$$

where R, θ can take on the values $R \geq 0$, $0 \leq \theta \leq 2\pi$.

Thus, $f_R(r) = \frac{R}{\sigma^2} e^{-\frac{1}{2}(R/\sigma)^2}$, $R \geq 0$

$f_\theta(\theta) = \frac{1}{2\pi}$, $0 \leq \theta \leq 2\pi$.

(4.24)

$y = g(x) = x^2$, $x = g^{-1}(y) = \pm\sqrt{y}$.

Define $g_1^{-1}(y) = +\sqrt{y}$, $g_2^{-1}(y) = -\sqrt{y}$.

$\frac{\partial g_1^{-1}(y)}{\partial y} = \frac{1}{2\sqrt{y}}$, $\frac{\partial g_2^{-1}(y)}{\partial y} = \frac{-1}{2\sqrt{y}}$. Then

$$f_Y(y) = f_X(x) \left|\frac{\partial g_1^{-1}(y)}{\partial y}\right| \Big|_{x = g_1^{-1}(y)} + f_X(x) \left|\frac{\partial g_2^{-1}(y)}{\partial y}\right| \Big|_{x = g_2^{-1}(y)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{2\sqrt{y}} e^{-\frac{1}{2}y/\sigma^2} + \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{2\sqrt{y}} e^{-\frac{1}{2}y/\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y}} e^{-\frac{1}{2}y/\sigma^2}, \quad y \geq 0.$$

(4.25)

First consider the random variable $z = ax$, $-\infty < x < \infty$.

Then $f_Z(z) = \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{1}{2}(z/a\sigma)^2}$, $-\infty < z < \infty$.

Now $y = \begin{cases} z, & z \geq 0 \\ 0, & z < 0 \end{cases}$ $\therefore P(y=0) = P(z \leq 0) = \frac{1}{2}$.

$$\therefore f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{1}{2}(z/a\sigma)^2}, \quad y \geq 0.$$

The δ-function appears in the p.d.f. of y because $x < 0$ maps into $y = 0$ so that $P(y=0)$ is finite.

4.26

(a) Note that the transformation is single valued and monotonic.

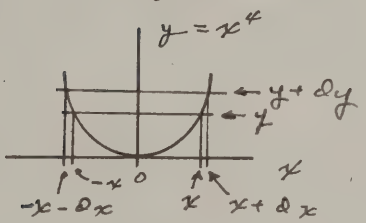
$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x = y^{1/3}}$$

$$\frac{dx}{dy} = \frac{1}{3} y^{-2/3}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \frac{1}{3} y^{-2/3} \Big|_{x = y^{1/3}}$$

$$= \frac{1}{3y^{2/3}\sqrt{2\pi}\,\sigma} e^{-y^{2/3}/2\sigma^2}$$

(b)



Note that $f_Y(y) = 0, \quad y < 0$

$$P\{y < Y \le y + dy\} = P\{-x - dx < X \le -x\} + P\{x < X \le x + dx\}$$

Since $f_X(x)$ is even

$$P\{-x - dx < X \le -x\} = P\{x < X \le x + dx\}$$

Thus

$$P\{y < Y \le y + dy\} = 2\,P\{x < X \le x + dx\}$$

$$f_Y(y) = 2 f_X(x) \left| \frac{dx}{dy} \right|_{x = y^{1/4}}$$

$$x = y^{1/4} \qquad \frac{dx}{dy} = \frac{1}{4} y^{-3/4}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{2y^{3/4}\sqrt{2\pi}\,\sigma} e^{-\sqrt{y}/2\sigma^2}, & y > 0 \\ 0, & y < 0 \end{cases}$$

(4.27)

(a) $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$ $\qquad \frac{dx}{dy} = \frac{1}{3y^{2/3}}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \frac{1}{3y^{4/3}} \Big|_{x=y^{1/3}}$$

$$= \frac{1}{3y^{2/3}\sqrt{2\pi}\sigma} e^{-(y^{1/3}-m)^2/2\sigma^2}$$

(b) with $y = x^4$ we have, for each value of y, two values of x, denoted x_1 and x_2. Clearly $x_1 = y^{1/4}$ and $x_2 = -y^{1/4}$.

$$P\{y < Y \le y + dy\} = P\{x_1 < X \le x_1 + dx_1\} + P\{x_2 < X \le x_2 + dx_2\}$$

Since $f_X(x)$ is not symmetrical about zero, the two terms on the right hand side of the above equation are <u>not</u> equal.

$$f_Y(y) = f_X(x_1) \left| \frac{dx}{dy} \right|_{x=x_1} + f_X(x_2) \left| \frac{dx}{dy} \right|_{x=x_2}$$

$$\frac{dx}{dy} = \frac{d}{dy}(y^{1/4}) = \frac{1}{4} y^{-3/4} = \frac{1}{4y^{3/4}}$$

with $x_1 = y^{1/4}$ and $x_2 = -y^{1/4}$ we have

$$f_Y(y) = \frac{1}{4y^{3/4}} \left[f_X(y^{1/4}) + f_X(-y^{1/4}) \right]$$

(4.27) (cont)

$$f_Y(y) = \frac{1}{4y^{3/4}\sqrt{2\pi}\sigma}\left[e^{-(y^{1/4}-m)^2/2\sigma^2} + e^{-(-y^{1/4}-m)^2/2\sigma^2}\right]$$

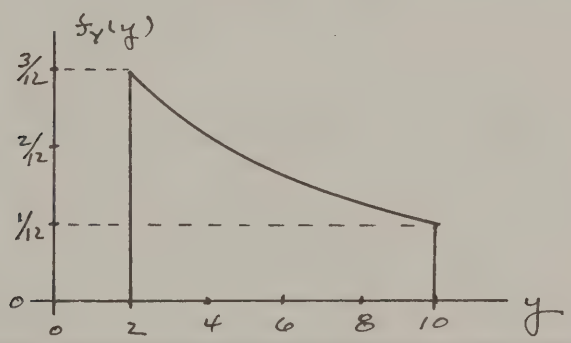
(4.28)

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|_{x=\sqrt{y-1}}$$

$$\frac{dx}{dy} = \frac{1}{2}(y-1)^{-1/2} = \frac{1}{2\sqrt{y-1}}$$

Thus

$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y-1}}, & 2 < y < 10 \\ 0, & \text{otherwise} \end{cases}$$



(4.29)

(a) $A > 0$ since $f_X(x) \geq 0$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = A\int_{-\infty}^{\infty} e^{-b|x|}\,dx = 2A\int_0^{\infty} e^{-bx}\,dx$$

$$= 2A\left(\frac{-1}{b}\right)e^{-bx}\Big|_0^{\infty} = 2\frac{A}{b}$$

Thus $2A = b$ Note that $b > 0$

(b) $E\{X\} = \frac{b}{2}\int_{-\infty}^{\infty} x e^{-b|x|}\,dx$

Note that the integrand is odd. Thus $E\{X\} = 0$

(4.29) (Cont)

Since $E\{x\} = 0$, $E\{x^2\} = \sigma_x^2$

$$E\{x^2\} = \sigma_x^2 = \int_{-\infty}^{\infty} \frac{b}{2} x^2 e^{-b|x|}\, dx$$

$$= b\int_0^{\infty} x^2 e^{-bx}\, dx = b\int_0^{\infty} \frac{y^2}{b^2} e^{-y}\, \frac{dy}{b}$$

$$= \frac{1}{b^2}\int_0^{\infty} y^2 e^{-y}\, dy = \frac{2!}{b^2} = \frac{2}{b^2}$$

(4.30)

$$f_x(x) = A e^{-bx} u(x)$$

(a) $$1 = \int_0^{\infty} A e^{-bx}\, dx = A\left(\frac{-1}{b}\right) e^{-bx}\Big|_0^{\infty} = \frac{A}{b}$$

Thus $A = b$

(b) $$F_x(x) = b\int_0^{x} e^{-b\lambda}\, d\lambda = b\left(\frac{-1}{b}\right) e^{-b\lambda}\Big|_0^{x}$$

$F_x(x) = 1 - e^{-bx}$, $x \geq 0$

$F_x(x) = 0$, $x < 0$

(c) $$E\{x^2\} = b\int_0^{\infty} x^2 e^{-bx}\, dx = b\int_0^{\infty} \frac{y^2}{b^2} e^{-y}\, \frac{dy}{b}$$

$$= \frac{1}{b^2}\int_0^{\infty} y^2 e^{-y}\, dy = \frac{1}{b^2}\Gamma(3) = \frac{2}{b^2}$$

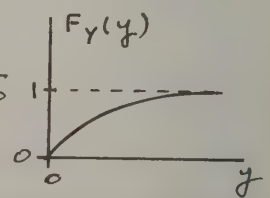
(d) To determine the variance of x we need $E\{x\}$

$$E\{x\} = b\int_0^{\infty} x e^{-bx}\, dx$$

$$= b\int_0^{\infty} \frac{y}{b} e^{-y}\, \frac{dy}{b}$$

$$= \frac{1}{b}\int_0^{\infty} y e^{-y}\, dy = \frac{1}{b}$$

$$\sigma_x^2 = E\{x^2\} - E^2\{x\} = \frac{1}{b^2}$$



4.31 We can use the results of the previous problem with $b=4$. In the previous problem we found $F_x(x) = [1-e^{-bx}]u(x)$ and $\sigma_x^2 = \frac{1}{b^2}$.

(a) $F_x(0.4) = 1-e^{-4(0.4)} = 1-e^{-1.6} = 0.7981$

(b) $P\{0.4 < x \le 0.8\} = (1-e^{-4(0.4)}) - (1-e^{-4(0.8)})$
$= e^{-3.2} - e^{-1.6} = 0.1611$

(c) $\sigma_x^2 = \frac{1}{b^2} = \frac{1}{(4)^2} = \frac{1}{16} = 0.0625$

4.32 (a) $E\{x^{2N}\} = \int_{-\infty}^{\infty} x^{2N} f_x(x)\,dx = \int_{-\infty}^{\infty} x^{2N} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}\,dx$

with $y = x/\sqrt{2}\sigma$ we have

$E\{x^{2N}\} = 2\int_0^{\infty} (y\sqrt{2}\sigma)^{2N} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2} \sqrt{2}\sigma\,dy$
$= \frac{1}{\sqrt{\pi}} 2^{N+1} \sigma^{2N} \int_0^{\infty} y^{2N} e^{-y}\,dy$

It can be shown that $\int_0^{\infty} y^{2N} e^{-y}\,dy = \frac{1\cdot 3\cdot 5 \cdots (2N-1)}{2^{N+1}}\sqrt{\pi}$

Thus $E\{x^{2N}\} = 1\cdot 3\cdot 5 \cdots (2N-1)\,\sigma^{2N}$

(b) $E\{x^{2N-1}\} = \int_{-\infty}^{\infty} \underbrace{x^{2N-1}}_{\text{odd}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}}_{\text{even fcn}}\,dx$

Since the integrand is odd $E\{x^{2N-1}\} = 0$

4.33

$$f_{XY}(x,y) = \begin{cases} 2e^{-(2x+y)} & x,y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) $E\{X^M Y^N\} = E\{X^M\}E\{Y^N\}$ by independence

$E\{X^M\} = \int_0^{\infty} x^M 2e^{-2x}\,dx = \frac{2M!}{2^{M-1}} = M!\,2^{-M}$

$E\{Y^N\} = \int_0^{\infty} y^N e^{-y}\,dy = N!$

Thus $E\{X^M Y^N\} = M!\,N!\,2^{-M}$

(4.33) (Cont.)

(b) $\mu_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})(y-\bar{y})\,f_{XY}(x,y)\,dx\,dy.$

From part (a): $\bar{x} = 1/2$, $\bar{y} = 1$. Thus

$\mu_{XY} = \int_0^{\infty}\int_0^{\infty}(x-1/2)(y-1)\,2e^{-(2x+y)}\,dx\,dy$

$= 2\int_0^{\infty}\int_0^{\infty}[xy - x - 1/2\,y + 1/2]\,e^{-2x}e^{-y}\,dx\,dy$

$= 2\int_0^{\infty} y e^{-y}\,dy \int_0^{\infty}(x-1/2)e^{-2x}\,dx + 2\int_0^{\infty}e^{-y}\,dy\int_0^{\infty}(1/2-x)e^{-2x}\,dx$

$= 5/2\int_0^{\infty} y e^{-y}\,dy - 5/2\int_0^{\infty}e^{-y}\,dy = 5/2 - 5/2 = 0.$

(4.34)

$f_X(x) = \frac{x}{\sigma^2}e^{-1/2(x/\sigma)^2}$, $x \geq 0$.

$E\{X\} = \frac{1}{\sigma^2}\int_0^{\infty}x^2 e^{-1/2(x/\sigma)^2}\,dx = \sqrt{\frac{\pi}{2}}\,\sigma$ from integral tables.

$E\{X^2\} = \frac{1}{\sigma^2}\int_0^{\infty}x^3 e^{-1/2(x/\sigma)^2}\,dx$. Using change of variables with $y = x^2$ gives

$E\{X^2\} = \frac{1}{2\sigma^2}\int_0^{\infty} y\, e^{-\frac{1}{2\sigma^2}y}\,dy = 2\sigma^2$ from integral tables.

$E\{(X-\bar{X})^2\} = E\{X^2\} - E^2\{X\} = 1/2(4-\pi)\sigma^2.$

(4.35)

$E\{Z\} = E\{X+R\} = E\{X\} + E\{R\}.$

From prob. 4.29, $E\{X\} = 0$. From prob. 4.34, $E\{R\} = \sqrt{\frac{\pi}{2}}\,\sigma$.

$\therefore E\{Z\} = \sqrt{\frac{\pi}{2}}\,\sigma.$

(4.36)

(a) $E\{Z\} = E\{X_1 + X_2 + \ldots + X_N\} = E\{X_1\} + E\{X_2\} + \ldots + E\{X_N\} = NM.$

(b) $E\{Z^2\} = E\{(X_1 + X_2 + \ldots + X_N)(X_1 + X_2 + \ldots + X_N)\}$

$= E\{\sum_{i=1}^{N} X_1 X_i\} + E\{\sum_{i=1}^{N} X_2 X_i\} + \ldots + E\{\sum_{i=1}^{N} X_N X_i\}$

$= (1+P) + \underbrace{(P+1+P) + \ldots + (P+1+P)}_{N-2 \text{ of these}} + (P+1)$

(4.36) (Cont.)

$\therefore E\{Z^2\} = 2+2P + (N-2)(1+2P) = N+(N-1)2P.$

$VAR(Z) = E\{Z^2\} - E^2\{Z\} = N+2(N-1)P - (NM)^2.$

(4.37)

Consider the random variables U and V where U, V are independent Gaussian with zero mean and variance σ^2. Define X, Y as

$X = PU + \sqrt{1-\rho^2}\,V = g_1(U,V)$, $Y = U = g_2(U,V)$.

Note that $Var(X) = Var(Y) = \sigma^2$ and X, Y are Gaussian. The joint density $f_{XY}(x,y)$ can be found:

$u = g_1^{-1}(x,y) = y$, $v = g_2^{-1}(x,y) = \frac{1}{\sqrt{1-\rho^2}}[x - py]$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{\sqrt{1-\rho^2}} & \frac{\rho}{\sqrt{1-\rho^2}} \end{vmatrix} = \frac{1}{\sqrt{1-\rho^2}}.$$

$$\therefore f_{XY}(x,y) = \frac{1}{\sqrt{1-\rho^2}} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(u^2+v^2)} \Big|_{\substack{u = g_1^{-1}(x,y) \\ v = g_2^{-1}(x,y)}}$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma^2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}.$$

Thus, X, Y are equivalent to the random variables in problem statement.

$E\{XY\} = E\{(PU + \sqrt{1-\rho^2}\,V)U\} = P\sigma^2.$

$\therefore$ P = correlation coefficient.

(4.38) See the solution to problem 4.54. The purpose of Problem 4.38 is to calculate the condition density and to calculate conditional mean and variance. The purpose of Problem 4.54 is to reentorce a fundamental property of a Gaussian RV.

(4.39)

(a) $M_x(jv) = E\{e^{jvx}\} = \int_{-\infty}^{\infty} e^{jvx}\frac{b}{2}e^{-b|x|}dx$

$= \int_{-\infty}^{0} e^{jvx}\frac{b}{2}e^{bx}dx + \int_{0}^{\infty} e^{jvx}\frac{b}{2}e^{-bx}dx$

$= \frac{b}{2}\frac{1}{jv+b}e^{(jv+b)x}\Big|_{-\infty}^{0} + \frac{b}{2}\frac{1}{jv-b}e^{(jv-b)x}\Big|_{0}^{\infty} = \frac{b^2}{v^2+b^2}$.

(b) $E\{X\} = -j\left.\frac{\partial M_x}{\partial v}\right|_{v=0} = -jb^2(b^2+v^2)^{-2}2v\Big|_{v=0} = 0.$

$E\{X^2\} = (-j)^2\left.\frac{\partial^2 M_x}{\partial v^2}\right|_{v=0} = 2b^2(b^2+v^2)^{-2}\Big|_{v=0} + 2b^2v(-3)(b^2+v^2)^{-4}2v\Big|_{v=0}$

$= 2/b^2$

(c) of course the results agree with those of Prob. 4.29.

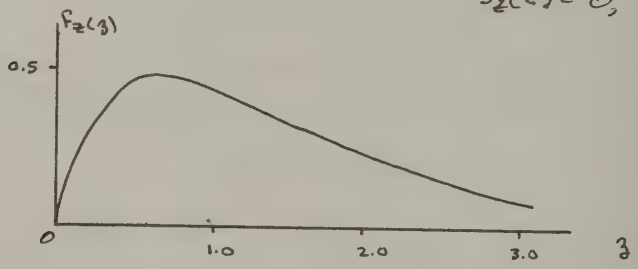
(4.40)

$f_z(z) = f_x(x) * f_y(y) = \int_{-\infty}^{\infty} f_x(z-\tau)f_y(\tau)d\tau.$

$f_x(x) = e^{-x}u(x)$, $f_y(y) = 2e^{-2y}u(y)$

$\therefore f_z(z) = \int_0^z e^{-(z-\tau)}2e^{-2\tau}d\tau = 2e^{-z}\int_0^z e^{-\tau}d\tau = 2e^{-z}(1-e^{-z}),\ z>0$

$f_z(z) = 0,\ z \le 0$

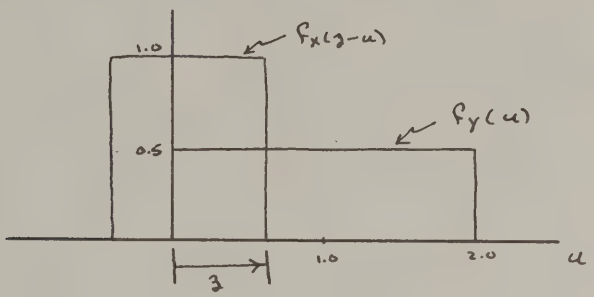


(4.41)

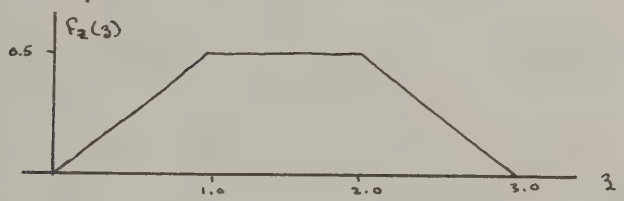
$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{, otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1/2 & , 0 \le y \le 2 \\ 0 & \text{, otherwise} \end{cases}$$

$Z = X + Y$

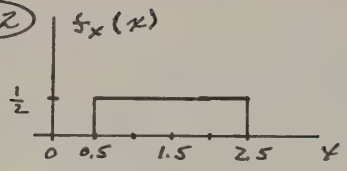
$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(z-u)\, f_Y(u)\, du$. Graphically:

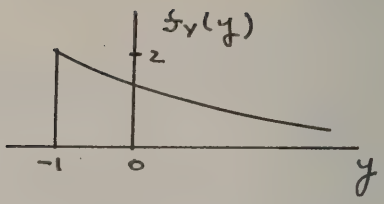


The resulting p.d.f. is shown below.

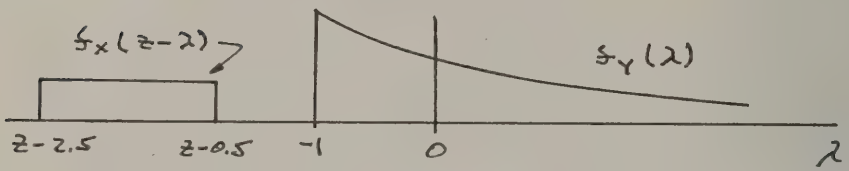


4.42





$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(\lambda)\, f_X(z-\lambda)\, d\lambda$$



There are three cases to consider:

Case I: $z - 0.5 \le -1$

$z \le -0.5$ $\qquad f_Z(z) = 0$

Case II: $z - 0.5 \le -1$ and $z - 2.5 \le -1$

$z \le -0.5$ $\qquad z \le 1.5$

$$f_Z(z) = \int_{-1}^{z-0.5} \frac{1}{2}(2)\, e^{-2(\lambda+1)}\, d\lambda$$

$$= e^{-2} \int_{-1}^{z-0.5} e^{-2\lambda}\, d\lambda$$

$$= e^{-2}\left(\frac{-1}{2}\right)\left(e^{-2(z-0.5)} - e^{2}\right)$$

$$= \frac{1}{2}\left(1 - e^{-2(z+0.5)}\right), \quad -0.5 \le z \le 1.5$$

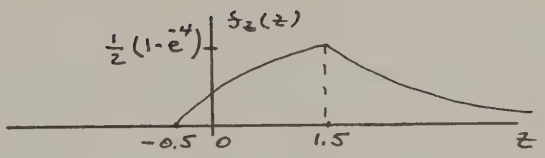
Case III: $z - 2.5 \ge -1$ $\qquad z \ge 1.5$

$$f_Z(z) = \int_{z-0.5}^{z-2.5} e^{-2(\lambda+1)}\, d\lambda$$

$$= e^{-2}\left(\frac{-1}{2}\right)\left(e^{-2(z-2.5)} - e^{-2(z-0.5)}\right)$$

$$= \frac{1}{2}\left(e^{-2(z-1.5)} - e^{-2(z+0.5)}\right), \quad z \ge 1.5$$

(4.42) (Cont)



To determine $E\{z\}$ it is easiest to note that $E\{z\} = E\{x\} + E\{y\}$

$$E\{x\} = \int_{0.5}^{2.5} \frac{1}{2} x \, dx = \frac{1}{4}\left[(2.5)^2 - (0.5)^2\right] = 1.5$$

$$E\{y\} = \int_{-1}^{\infty} 2e^{-2(y+1)} \, dy = 2e^{-2} \int_{-1}^{\infty} y e^{-2y} dy$$

$$= (2e^{-2}) \frac{e^{-2y}}{4}(-2y-1)\Big|_{-1}^{\infty} = -\frac{1}{2}$$

Thus

$$E\{z\} = 1.5 - 0.5 = 1.0$$

(4.43)

$$f_x(x) = \frac{1}{\sqrt{16\pi}} e^{-x^2/16} = \frac{1}{\sqrt{2\pi}\sqrt{8}} e^{-x^2/2(8)}$$

Thus x is zero-mean Gaussian with $\sigma_x^2 = 8$.

(a) $E\{x\} = 0$ $\quad E\{x^2\} = \sigma_x^2 = 8$

(b) $f_y(y) = f_x(x) \left| \frac{dx}{dy} \right|_{x = \frac{y-2}{3}}$ $\quad \frac{dx}{dy} = \frac{1}{3}$

Thus

$$f_y(y) = \frac{1}{3} \frac{1}{\sqrt{2\pi}\sqrt{8}} e^{-(y-2)^2/2(9)(8)}$$

$$= \frac{1}{\sqrt{2\pi(72)}} e^{-(y-2)^2/2(72)}$$

(4.43)

(c & d) By inspection of $f_Y(y)$

$$E\{Y\} = 2, \quad \sigma_y^2 = 72$$
$$E\{Y^2\} = \sigma_y^2 + E^2\{Y\} = 72 + 4 = 76$$

or

$$E\{Y\} = E\{2 + 3X\} = E\{2\} + 3E\{X\}$$
$$= 2 + 0 = 2$$
$$E\{Y^2\} = E\{4 + 12X + 9X^2\}$$
$$= E\{4\} + 12E\{X\} + 9E\{X^2\}$$
$$= 4 + 12(0) + 9(8)$$
$$= 4 + 72 = 76$$
$$\sigma_y^2 = E\{Y^2\} - E^2\{Y\}$$
$$= 76 - 4 = 72$$

(4.44)

(a) $$M_x(jv) = \int_{-\infty}^{\infty} f_x(x) e^{jvx} dx = \frac{1}{b-a} \int_a^b e^{jvx} dx$$

$$= \frac{1}{b-a} \frac{1}{jv} \left(e^{jvb} - e^{jva} \right)$$

If desired, $M_x(jv)$ can be placed in the form of a sinc function.

(b)
$$E\{x\} = (-j) \left. \frac{\partial M_x(jv)}{\partial v} \right|_{v=0}$$
$$= -j \frac{1}{j(b-a)} \left[\frac{v(jbe^{jbv} - jae^{jva}) - (e^{jvb} - e^{jva})}{v^2} \right]_{v=0}$$

Note that both the numerator and the denominator $\to 0$ as $v \to 0$. Applying

(4.44) (Cont.)

L'Hospital's rule yields

$$E\{X\} = \frac{1}{a-b}\,\frac{1}{2v}\left[v(-b^2)e^{jvb} + jbe^{jvb} + va^2e^{jva} - jae^{jva} - jbe^{jvb} + jae^{jva}\right]_{v=0}$$

$$E\{x\} = \frac{1}{2(a-b)}(a^2-b^2) = \frac{(a+b)(a-b)}{2(a-b)} = \frac{1}{2}(a+b)$$

(4.45)

(a) $$E\{X\} = \int_{-\infty}^{\infty} f_x(x)e^{jvx}\,dx = \int_0^{\infty} ae^{-ax}e^{jvx}\,dx$$

$$= a\int_0^{\infty} e^{-(a-jv)x}\,dx = \frac{a}{a-jv}$$

(b) $$\frac{dM_x(jv)}{dv} = \frac{-a(-j)}{(a-jv)^2} = \frac{ja}{(a-jv)^2}$$

$$\frac{d^2M_x(jv)}{dv^2} = \frac{-ja(2)(a-jv)(-j)}{(a-jv)^4} = \frac{-2a}{(a-jv)^3}$$

$$E\{X\} = (-j)\left.\frac{dM_x(jv)}{dv}\right|_{v=0} = -j\left.\frac{ja}{(a-jv)^2}\right|_{v=0} = \frac{a}{a^2} = \frac{1}{a}$$

$$E\{X^2\} = (-j)^2\left.\frac{dM_x(jv)}{dv}\right|_{v=0} = \left.\frac{2a}{(a-jv)^3}\right|_{v=0} = \frac{2a}{a^3} = \frac{2}{a^2}$$

Check:

$$E\{X^n\} = \int_{-\infty}^{\infty} x^n f_x(x)\,dx = \int_0^{\infty} x^n ae^{-ax}\,dx$$

with $y = ax$ we have

$$E\{X^n\} = a\int_0^{\infty} \frac{y^n}{a^n}e^{-y}\frac{dy}{a} = \frac{1}{a^n}\int_0^{\infty} y^n e^{-y}\,dy$$

$$= \frac{1}{a^n}\Gamma(n+1) = \frac{n!}{a^n}$$

Thus

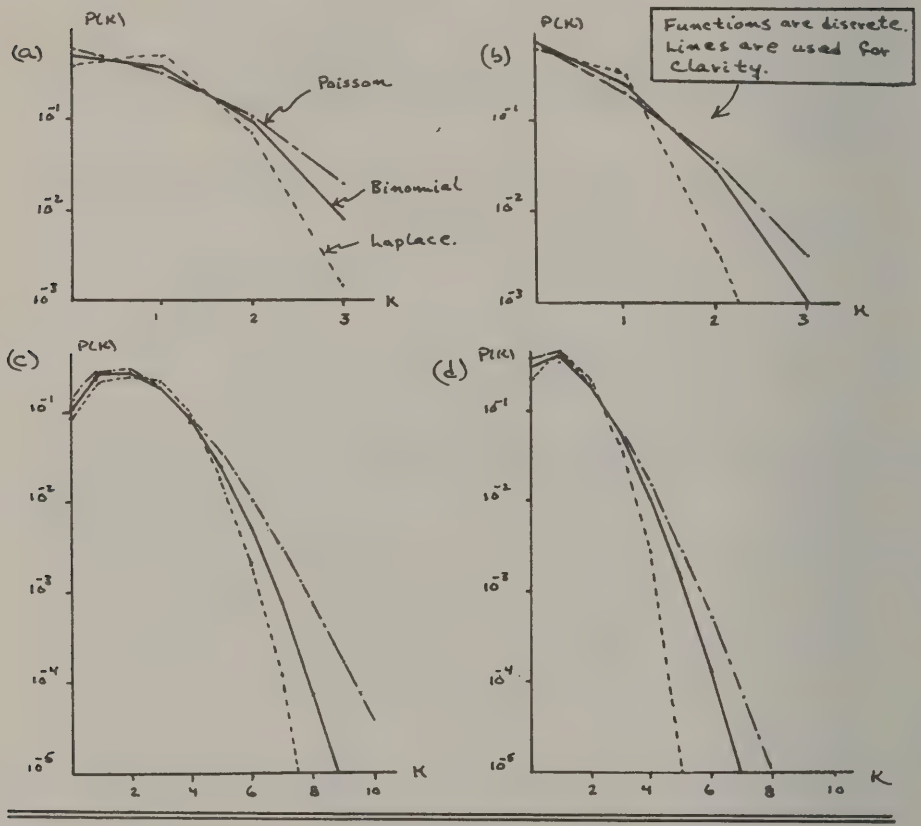
$$E\{X\} = \frac{1}{a} \quad \text{and} \quad E\{X^n\} = \frac{2}{a^2}$$

(4.46)

Binomial : $P(K) = \binom{n}{K} p^K (1-p)^{n-K}$

Laplace approx.: $P(K) = \frac{1}{\sqrt{2\pi np(1-p)}} e^{-(K-np)^2/2np(1-p)}$

Poisson approx: $P(K) = \frac{(np)^K}{K!} e^{-np}$ where $K = 0, 1, 2, \ldots n$.



(4.47)

$$P\{3 \text{ or } 4 \text{ heads in } 8 \text{ tosses}\} = \sum_{k=3}^{4} P_8(k)$$

$$= \sum_{k=3}^{4} \binom{8}{k} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{8-k} = \left[\frac{8!}{3!\,5!} + \frac{8!}{4!\,4!}\right] \left(\tfrac{1}{2}\right)^8$$

$$= \left[56 + 70\right] \frac{1}{256} = \frac{120}{256} = 0.4688$$

(4.48)

$$n = 80, \quad p = q = \tfrac{1}{2}, \quad npq = 80\left(\tfrac{1}{2}\right)^2 = 20$$

$$np = 80\left(\tfrac{1}{2}\right) = 40, \qquad m = 40, \qquad \sigma^2 = 20$$

$$P\{30 < K \le 40\} = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{30}^{40} e^{-(x-m)^2/2\sigma^2}\, dx$$

with $\lambda = \dfrac{x-m}{\sqrt{2}\,\sigma}$

$$P\{30 < K \le 40\} = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-10/\sqrt{2}\sqrt{20}}^{0} e^{-\lambda^2} \sigma\sqrt{2}\, d\lambda$$

$$= \frac{1}{\sqrt{\pi}} \int_{-10/\sqrt{40}}^{0} e^{-\lambda^2} d\lambda = \frac{1}{\sqrt{\pi}} \int_{0}^{10/\sqrt{40}} e^{-\lambda^2} d\lambda$$

$$= \frac{1}{2}\, erf\left(\frac{10}{\sqrt{40}}\right) = \frac{1}{2}\, erf\left(\sqrt{\tfrac{5}{2}}\right) = 0.488$$

(4.49)

(a)

$$N_p = \frac{26!}{(26-4)!} = (26)(25)(24)(23)$$

$$= 358{,}000$$

(b)

$$N_p = (26)^4 = 456{,}976$$

4.50

(a) $P_n(k) = \dfrac{n!}{k!(n-k)!} p^k q^{n-k}$ $\qquad p = q = \frac{1}{2},\ n = 10$

$P\{\text{fewer than 3 heads in 10 tosses}\}$

$$= \sum_{k=0}^{2} \frac{10!}{k!(10-k)!} \left(\frac{1}{2}\right)^{10}$$

$$= \left[\frac{10!}{0!\,10!} + \frac{10!}{1!\,9!} + \frac{10!}{2!\,8!}\right]\left(\frac{1}{2}\right)^{10}$$

$$= \left[1 + 10 + 45\right]\left(\frac{1}{2}\right)^{10} = \frac{56}{1024}$$

$$= 0.0547$$

(b) $P_n(k) \approx \dfrac{1}{\sqrt{2\pi npq}}\, e^{-(k-np)^2/2npq}$

$$npq = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{5}{2}$$

$P\{\text{fewer than 3 heads in 10 tosses}\}$

$$= \sum_{k=0}^{2} \frac{1}{\sqrt{5\pi}}\, e^{-(k-5)^2/5}$$

$$= \frac{1}{\sqrt{5\pi}}\left[e^{-5} + e^{-16/5} + e^{-9/5}\right]$$

$$= 0.0537$$

(c) Percent Error $= \dfrac{0.0547 - 0.0537}{0.0547}(10^2)$

$$= 1.84\%$$

(4.51)

This is binomial with $P=10^{-6}$, $N=10^6$. Approximate using Poisson.

$$P_n(k) \approx \frac{(NP)^k}{k!} e^{-NP} = \frac{.368}{k!}$$

$P_n(0) = .368$, $P_n(1) = .368$, $P_n(2) = .184$

P(Two or less errors) = .9197 and

P(More than two errors) = 1 - .9197 = .0803.

(4.52)

(a) Note that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2}\,du = \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(u-b)^2}\,du = \sqrt{\frac{2\pi}{a}}$ for $a>0$.

Let $u = (x-m_x)/\sigma_x$, $v = (y-m_y)/\sigma_y$, $K = [2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}]^{-1}$.

Then $f_{XY}(x,y) = K \exp\left[-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right]$.

Completing the square on v gives

$$f_{XY}(x,y) = K \exp\left[-\frac{1}{2(1-\rho^2)}(v-2\rho u)^2\right] \exp\left[-\tfrac{1}{2}u^2\right].$$

Substituting $v = (y-m_y)/\sigma_y$ gives

$$f_{XY}(x,y) = K \exp(-\tfrac{1}{2}u^2) \exp\left[\frac{-1}{2(1-\rho^2)\sigma_y^2}(y-m_y-2\rho u\sigma_y)^2\right].$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = K\exp(-\tfrac{1}{2}u^2)\int_{-\infty}^{\infty}\exp\left[\frac{-1}{2(1-\rho^2)\sigma_y^2}(y-m_y-2\rho u\sigma_y)^2\right]dy$$

$$= \sqrt{(1-\rho^2)\sigma_y^2\,2\pi}\;K\exp(-\tfrac{1}{2}u^2) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\,e^{-\frac{1}{2}\frac{(x-m_x)^2}{\sigma_x^2}}$$

(4.52) (Cont.)

The p.d.f. of y can be found in a similar fashion.

(b) First find the moment generating function of x.

$$M_x(jv) = E\{e^{jvx}\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{jvx} e^{-\frac{1}{2}\left(\frac{x-m_x}{\sigma_x}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[jvx - \tfrac{1}{2}x^2/\sigma_x^2 + \frac{xm_x}{\sigma_x^2} - \tfrac{1}{2}\frac{m_x^2}{\sigma_x^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2}m_x^2/\sigma_x^2} \exp\left[-\frac{1}{2\sigma_x^2}(x^2 - 2m_x x - 2jvx\sigma_x^2)\right] dx$$

Completing the square on x gives

$$M_x(jv) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\tfrac{1}{2}\sigma_x^2 v^2 + jm_x v\right] \exp\left[-\frac{1}{2\sigma_x^2}(x - m_x - jv\sigma_x^2)^2\right] dx$$

Using $\int_{-\infty}^{\infty} e^{-\frac{1}{2}a(u-b)^2} du = \sqrt{\frac{2\pi}{a}}$ gives

$$M_x(jv) = \exp\left[jvm_x - \tfrac{1}{2}\sigma_x^2 v^2\right].$$

$$E\{x\} = -j\, M_x'(jv)\Big|_{v=0} = m_x$$

$$E\{x^2\} = (-j)^2 M_x''(jv)\Big|_{v=0} = \sigma_x^2 + m_x^2$$

$$\mathrm{Var}(x) = E\{x^2\} - E^2\{x\} = \sigma_x^2.$$

The same procedure can be used to find the statistics of y.

(4.53)

(a) $1 = \int_{-\infty}^{\infty} f_x(x)\,dx = \int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = K \tan^{-1} x\Big|_{-\infty}^{\infty} = K\pi \Rightarrow K = \frac{1}{\pi}$.

(b) $E\{x\} = \int_{-\infty}^{\infty} \frac{x}{\pi} \frac{1}{1+x^2} dx = 0$ because integrand is odd.

$$E\{x^2\} = \int_{-\infty}^{\infty} \frac{x^2}{\pi} \cdot \frac{1}{1+x^2} dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^{-2}} dx = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+x^{-2}} dx$$

$$= \frac{2}{\pi}\int_0^{1} \frac{1}{1+x^{-2}} dx + \frac{2}{\pi}\int_1^{\infty} \frac{1}{1+x^{-2}} dx > \frac{2}{\pi} + \frac{2}{\pi}\int_1^{\infty} \frac{1}{2} dx \Rightarrow \infty.$$

(c) $M_x(jv) = E\{e^{jvx}\} = \int_{-\infty}^{\infty} e^{jvx} \frac{1}{\pi} \frac{1}{1+x^2} dx$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\cos vx}{1+x^2} dx + \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{j\sin vx}{1+x^2} dx = \frac{2}{\pi}\int_0^{\infty} \frac{\cos vx}{1+x^2} dx$$

$$= e^{-|v|} \quad \text{(from integral tables).}$$

(d) $f_z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M_z(jv)\, e^{-jvz} dv = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jvz} e^{-N|v|} dv$

$$= \frac{1}{\pi}\int_0^{\infty} \cos vz \cdot e^{-N|v|} = \frac{1}{\pi}\frac{N}{N^2+z^2}.$$

(4.54)

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{\sqrt{2\pi}\,\sigma_y \exp\left[\frac{-1}{2(1-\rho^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right\}\right]}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\,\exp\left[-\frac{1}{2}\frac{(y-m_y)^2}{\sigma_y^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}}\exp\left[\frac{-1}{2(1-\rho^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{\rho^2(y-m_y)^2}{\sigma_y^2}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}}\exp\left[\frac{-1}{2(1-\rho^2)\sigma_x^2}\left\{x^2 - \left[2m_x + \frac{2\rho\sigma_x}{\sigma_y}(y-m_y)\right]x + m_x^2 + \frac{2\rho m_x\sigma_x(y-m_x)}{\sigma_y} + \frac{\rho^2\sigma_x^2(y-m_y)^2}{\sigma_x^2}\right\}\right]$$

Completing the square on x gives

$$f_{X/Y}(x/y) = \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}}\exp\left[\frac{-1}{2(1-\rho^2)\sigma_x^2}\left\{x - m_x - \frac{\rho\sigma_x}{\sigma_y}(y-m_y)\right\}^2\right]$$

$\therefore$ The distribution is Gaussian.

$$E\{X/Y\} = m_x + \frac{\rho\sigma_x}{\sigma_y}(y-m_y)$$

$$\mathrm{Var}(X/Y) = \sigma_x^2(1-\rho^2).$$

(4.55)

(a) $M_y(jv) = E\{e^{jvy}\} = E\{e^{jv\sum_{i=1}^{n} x_i^2}\} = E\left\{\prod_{i=1}^{n} e^{jvx_i^2}\right\}$

$$= \prod_{i=1}^{N} E\{e^{jvx_i^2}\}.$$

$$E\{e^{jvx_i^2}\} = \int_{-\infty}^{\infty} e^{jvx^2}\frac{1}{\sqrt{2\pi}\sigma^2}e^{-\frac{1}{2}x^2/\sigma^2}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma^2}e^{-\left(\frac{1}{2\sigma^2}-jv\right)x^2}\,dx$$

$$= \frac{1}{\sigma\sqrt{\frac{1}{\sigma^2}-2jv}}\underbrace{\int_{-\infty}^{\infty}\frac{\sqrt{\frac{1}{\sigma^2}-2jv}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}-2jv\right)x^2}\,dx}_{1.0}$$

$$= \frac{1}{\sigma}\frac{1}{\sqrt{\frac{1}{\sigma^2}-2jv}} = (1-j2v\sigma^2)^{-1/2}.$$

$$M_y(jv) = \prod_{i=1}^{N} E\{e^{jvx_i^2}\} = \prod_{i=1}^{N}(1-j2v\sigma^2)^{-1/2} = (1-j2v\sigma^2)^{-N/2}.$$

(4.55) (Cont.)

(b) $(1-\alpha jv)^{-N/2} \Leftrightarrow \dfrac{y^{N/2-1}\, e^{-y/\alpha}}{\alpha^{N/2}\,\Gamma(N/2)}$.

$M_Y(iv) = (1-j2v\sigma^2)^{-N/2}$. Letting $\alpha = 2\sigma^2$ gives

$f_Y(y) = \dfrac{y^{N/2-1}\, e^{-y/2\sigma^2}}{(2\sigma^2)^{N/2}\,\Gamma(N/2)}$.

Note, by functional relationship, $Y \geq 0$.

(c) $E\{Y\} = E\left\{\sum_{i=1}^{N} X_i^2\right\} = \sum_{i=1}^{N} E\{X_i^2\} = N\sigma^2$.

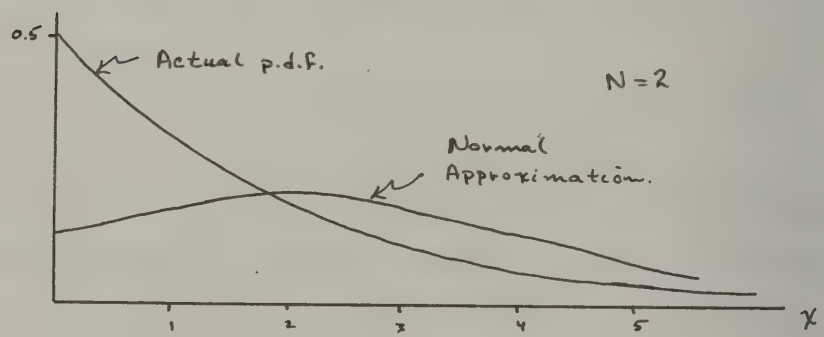
$E\{(X_i^2)^2\} = (-j)^2 \dfrac{\partial^2}{\partial v^2}(1-j2v\sigma^2)^{-1/2}\Big|_{v=0} = 3\sigma^4$.

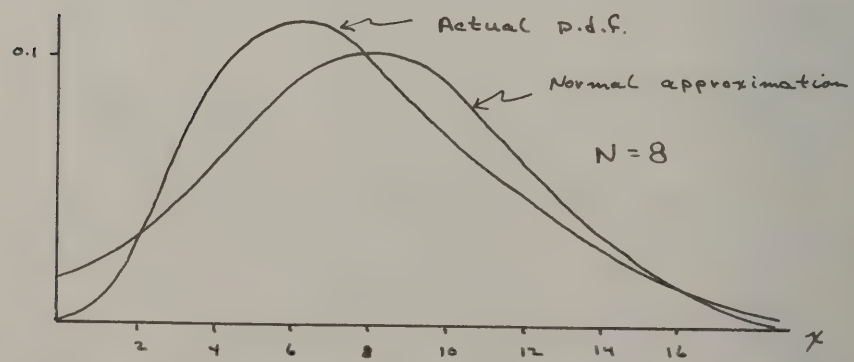
$\mathrm{Var}\{(X_i^2)^2\} = E\{(X_i^2)^2\} - E^2\{X_i^2\} = 3\sigma^4 - \sigma^4 = 2\sigma^4$.

$\mathrm{Var}\{Y\} = E\{Y^2\} - E^2\{Y\} = N\left[E\{(X_i)^2\} - E^2\{X_i^2\}\right] = 2N\sigma^4$.

Applying the Central Limit Theorem gives

$f_Y(y) = \dfrac{1}{\sqrt{4N\pi\sigma^4}}\, e^{-\frac{1}{2}(y-N\sigma^2)^2/2N\sigma^4}$.





Note improvement of approximation as $N \to \infty$. This is a consequence of the Central Limit Theorem.

(4.56)

$$f_X(x) = \frac{2m^m}{\Gamma(m)} x^{2m-1} e^{-mx^2}, \quad x \geq 0.$$

(a) For $m=1$, $f_X(x)$ reduces to $f_X(x) = 2x e^{-x^2}$, $x \geq 0$.

This is Rayleigh with $\sigma^2 = 1/2$.

(b) $E\{X^n\} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \int_0^{\infty} \frac{2m^m}{\Gamma(m)} x^{2m-1+n} e^{-mx^2} dx$

$= c \int_0^{\infty} x^K e^{-mx^2} dx$ where $c = \frac{2m^m}{\Gamma(m)}$, $K = 2m-1+n$.

From integral tables:

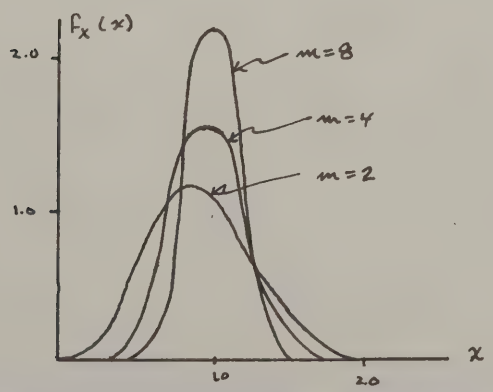
$$\int_0^{\infty} y^i e^{-ay^2} dy = \begin{cases} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (i-1)}{2^{(i+2)/2} a^{i/2}} \sqrt{\pi/a}, & \text{for } i \text{ odd} \\ \left(\frac{i-1}{2}\right)! / 2a^{(i+1)/2}, & \text{for } i \text{ even.} \end{cases}$$

$$\therefore E\{X^n\} = \begin{cases} c \cdot \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (K-1)}{2^{(K+2)/2} m^{K/2}} \sqrt{\frac{\pi}{m}}, & \text{for } K \text{ even.} \\ c \cdot \left(\frac{K-1}{2}\right)! / 2m^{(K+1)/2}, & \text{for } K \text{ odd.} \end{cases}$$

Note that K is odd when n is even, and K is even when n is odd. Thus

$$E\{X^n\} = \begin{cases} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2m+n-2)}{2^m (2m)^{(n-1)/2} \Gamma(m)} \sqrt{\frac{\pi}{m}}, & \text{for } n \text{ odd} \\ \frac{\Gamma(m+n/2)}{\Gamma(m) m^{n/2}}, & \text{for } n \text{ even} \end{cases}$$

(c)



(4.57)

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}(y-m_y)^2/\sigma_y^2} \qquad Y = \ln X.$$

(a) Since $X = e^Y$ and $\frac{dy}{dx} = \frac{1}{x}$ we have

$$f_X(x) = f_Y(y)\left|\frac{dy}{dx}\right|\Big|_{y=\ln x} = \frac{1}{\sqrt{2\pi}\sigma_y x} e^{-(\ln x - m_y)^2/2\sigma_y^2}.$$

Examination of functional relationship shows that $X \geq 0$.

(b) $$E\{X\} = \int_{-\infty}^{\infty} x f_Y(y)\,dy = \int_{-\infty}^{\infty} e^y \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}(y-m_y)^2/\sigma_y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} \exp\left[y - \frac{y^2}{2\sigma_y^2} + \frac{m_y}{\sigma_y^2}y - \frac{m_y^2}{2\sigma_y^2}\right] dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2\sigma_y^2}\left[y^2 - (2m_y + 2\sigma_y^2)y + m_y^2\right]\right\} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2\sigma_y^2}\left[y - (m_y + \sigma_y^2)\right]^2\right\} \cdot \exp\left[\tfrac{1}{2}\sigma_y^2 + m_y\right] dy$$

$$= \exp\left[m_y + \tfrac{1}{2}\sigma_y^2\right] \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2\sigma_y^2}(y - m_y - \sigma_y^2)^2} dy}_{1.0}$$

$$= \exp(m_y + \tfrac{1}{2}\sigma_y^2).$$

Using a similar technique, it follows that

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_Y(y)\,dy = \int_{-\infty}^{\infty} e^{2y} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}(y-m_y)^2/\sigma_y^2} dy$$

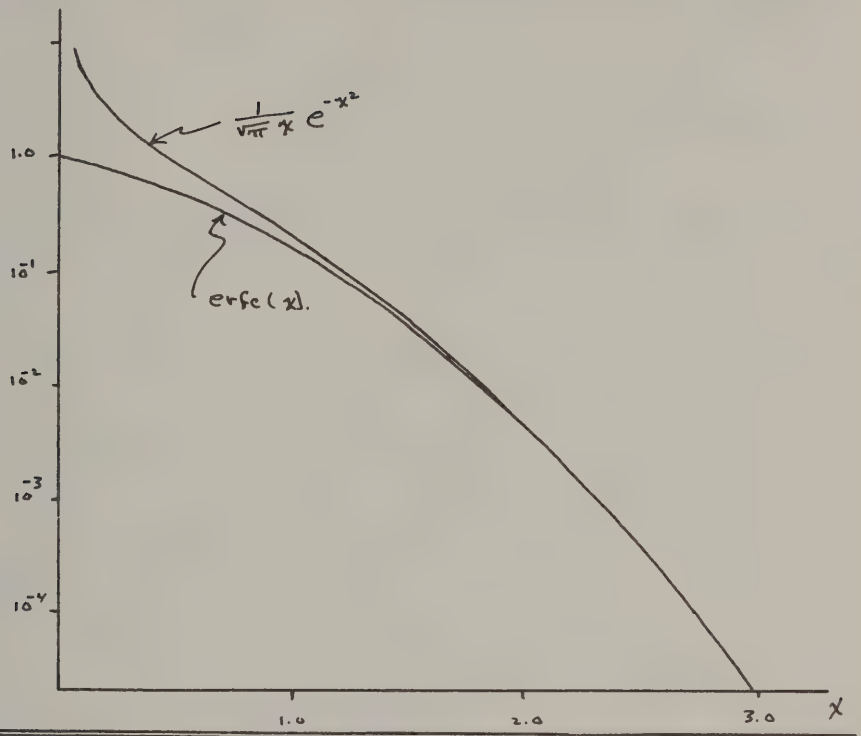
$$= \exp(2m_y + 2\sigma_y^2).$$

Thus

$$\mathrm{Var}(X^2) = E\{X^2\} - E^2\{X\}$$

$$= \exp(2m_y + 2\sigma_y^2) - \exp(2m_y + \sigma_y^2).$$

(4.58)



(4.59)

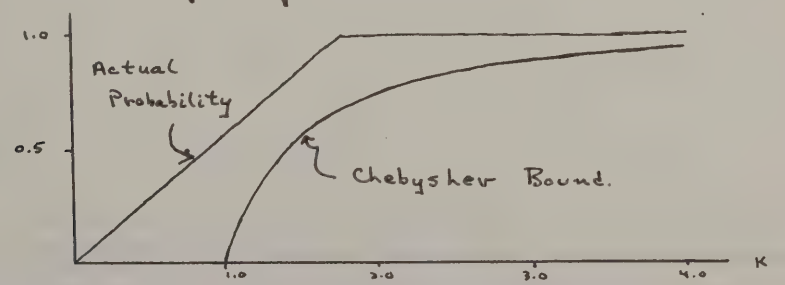
$f_X(x) = \begin{cases} \frac{1}{2}, & |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$ $\quad F_X(x) = \int_{-1}^{x} \frac{1}{2}\,du = \frac{1}{2}(1+x), \quad |x| \le 1.$

$m_X = 0, \quad \mathrm{Var}(X) = \frac{1}{3}.$

$\mathrm{Prob}(|X| \le c) = \mathrm{Prob}(-c \le X \le c) = F_X(c) - F_X(-c) = c.$

$\therefore \mathrm{Prob}(|x| \le K\sigma_X) = K\sqrt{1/3}.$

Chebyshev's inequality : $\mathrm{Prob}(|X - m_X| \le K\sigma_X) \ge 1 - \frac{1}{K^2}.$



(4.60) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy$

$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$

(a) $P\{X \le 11\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{10} e^{-(x-10)^2/2\sigma^2} dx + \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{11} e^{-(x-10)^2/2\sigma^2} dx$

where $\sigma = 5$

$P\{X \le 11\} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{11} e^{-(x-10)^2/2\sigma^2} dx$

Let $\lambda = \frac{x-10}{\sqrt{2}\sigma}$ we have

$P\{X \le 11\} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\sigma} \int_0^{1/\sqrt{2}\sigma} e^{-\lambda^2} \sqrt{2}\sigma\, d\lambda$

$= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{1/\sqrt{2}\sigma} e^{-\lambda^2} d\lambda$

$= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{1}{5\sqrt{2}}\right)$

(b) $P\{10 < X \le 12\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{12} e^{-(x-10)^2/2\sigma^2} dx$

$= \frac{1}{\sqrt{\pi}} \int_0^{2/\sqrt{2}\sigma} e^{-\lambda^2} d\lambda = \frac{1}{2} \operatorname{erf}\left(\frac{\sqrt{2}}{5}\right)$

(c) $P\{11 < X \le 13\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{13} e^{-(x-10)^2/2\sigma^2} dx - \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{11} e^{-(x-10)^2/2\sigma^2} dx$

$= \frac{1}{\sqrt{\pi}} \int_0^{3/\sqrt{2}\sigma} e^{-\lambda^2} d\lambda - \frac{1}{\sqrt{\pi}} \int_0^{1/\sqrt{2}\sigma} e^{-\lambda^2} d\lambda$

$= \frac{1}{2} \operatorname{erf}\left(\frac{3}{5\sqrt{2}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{1}{5\sqrt{2}}\right)$

(d) $P\{9 < X \le 10\} = \frac{1}{\sqrt{2\pi}\sigma} \int_9^{10} e^{-(x-10)^2/2\sigma^2} dx + \frac{1}{\sqrt{2\pi}\sigma} \int_{10}^{12} e^{-(x-10)^2/2\sigma^2} dx$

$= \frac{1}{2} \operatorname{erf}\left(\frac{1}{5\sqrt{2}}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{2}{5\sqrt{2}}\right)$

(4.61)

$$\sigma^2 \triangleq \int_{-\infty}^{\infty} (x-m)^2 f_x(x)\,dx \quad , \quad m = E\{x\}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-m)^2 f_x(x)\,dx \geq \int_{|x-m|>k\sigma} (x-m)^2 f_x(x)\,dx$$

where $\int_{|x-m|>k\sigma}$ denotes integration over only the region for which $|x-m|>k\sigma$

since $|x-m| > k\sigma$, $(x-m)^2 > k^2\sigma^2$, thus

$$\sigma^2 \geq \int_{|x-m|>k\sigma} (x-m)^2 f_x(x)\,dx \geq \int_{|x-m|>k\sigma} k^2\sigma^2 f_x(x)\,dx$$

thus $$\sigma^2 \geq k^2\sigma^2 \int_{|x-m|>k\sigma} f_x(x)\,dx$$

But $$\int_{|x-m|>k\sigma} f_x(x)\,dx = P\{|x-m|>k\sigma\}$$
$$= 1 - P\{|x-m|<k\sigma\}$$

so that

$$\sigma^2 \geq k^2\sigma^2 \{1 - P\{|x-m|<k\sigma\}\}$$

$$1 - P\{|x-m|<k\sigma\} \leq \frac{1}{k^2}$$

$$P\{|x-m|<k\sigma\} \geq 1 - \frac{1}{k^2}$$

(4.62)

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\,dy$$

Let $\lambda = y/\sqrt{2}$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x/\sqrt{2}}^{\infty} e^{-\lambda^2} \sqrt{2}\,d\lambda$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{2}}^{\infty} e^{-\lambda^2}\,d\lambda$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(x/\sqrt{2}\right)$$

In a similar manner $\mathrm{erfc}(x) = 2Q(\sqrt{2}x)$

(4.63)

$$P\{|X|>a\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-a} e^{-y^2/2\sigma^2}\,dy + \frac{1}{\sqrt{2\pi}\sigma}\int_{a}^{\infty} e^{-y^2/2\sigma^2}\,dy$$

$$= 2\,\frac{1}{\sqrt{2\pi}\sigma}\int_{a}^{\infty} e^{-y^2/2\sigma^2}\,dy$$

Let $\lambda = y/\sqrt{2}\sigma$. Then

$$P\{|X|>a\} = 2\,\frac{1}{\sqrt{2\pi}\sigma}\int_{a/\sqrt{2}\sigma}^{\infty} e^{-\lambda^2}\sqrt{2}\sigma\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_{a/\sqrt{2}\sigma}^{\infty} e^{-\lambda^2}\,d\lambda$$

$$= \operatorname{erfc}\left(\frac{a}{\sqrt{2}\sigma}\right)$$

with $a = k\sigma$ we have

$$P\{|X|>k\sigma\} = \operatorname{erfc}\left(\frac{k}{\sqrt{2}}\right)$$

(a) for $k=1$

$$P\{|X|>\sigma\} = \operatorname{erfc}\left(\frac{1}{\sqrt{2}}\right) = 0.3173$$

(b) For $k=2$

$$P\{|X|>2\sigma\} = \operatorname{erfc}\left(\frac{2}{\sqrt{2}}\right) = 0.0455$$

(c) For $k=3$

$$P\{|X|>3\sigma\} = \operatorname{erfc}\left(\frac{3}{\sqrt{2}}\right) = 0.0027$$

(4.64)

$$\sigma^2 = \frac{a}{2}\int_{-\infty}^{\infty} x^2 e^{-a|x|}\,dx = a\int_{0}^{\infty} x^2 e^{-ax}\,dx$$

with $y = ax$, $x = y/a$

$$\sigma^2 = a\int_{0}^{\infty} \frac{1}{a^2} y^2 e^{-y}\,\frac{1}{a}\,dy = \frac{1}{a^2}\int_{0}^{\infty} y^2 e^{-y}\,dy$$

(4.64) (cont.)

since $\int_0^\infty y^2 e^{-y}\,dy = \Gamma(3) = 2! = 2$ we have

$$\sigma^2 = \frac{2}{a^2} \quad \text{or} \quad a = \frac{\sqrt{2}}{\sigma}$$

$$P\{|X| > b\} = \frac{a}{2}\int_{-\infty}^{-b} e^{-a|x|}\,dx + \frac{a}{2}\int_b^\infty e^{-ax}\,dx$$

$$= a\int_b^\infty e^{-ax}\,dx = a\left(\frac{-1}{a}\right)\left(0 - e^{-ba}\right)$$

$$= e^{-ab}$$

with $b = K\sigma$ and $a = \frac{\sqrt{2}}{\sigma}$

$$P\{|X| > K\sigma\} = e^{-\sqrt{2}K}$$

(a) $P\{|X| > \sigma\} = e^{-\sqrt{2}} = 0.2431$

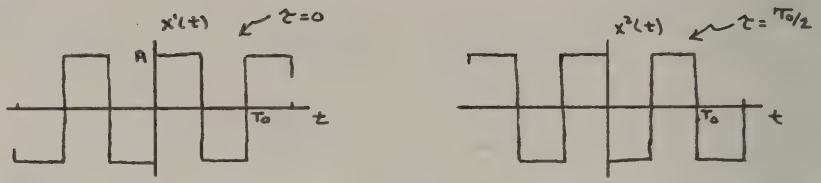
(b) $P\{|X| > 2\sigma\} = e^{-2\sqrt{2}} = 0.0591$

(c) $P\{|X| > 3\sigma\} = e^{-3\sqrt{2}} = 0.0144$

The results are greater for the Laplacian-amplitude pdf than for the Gaussian pdf. The Gaussian pdf "falls-off" faster than the Laplacian-amplitude pdf (e^{-x^2} for the Gaussian pdf and e^{-x} for the Laplacian-amplitude pdf.).

5.1

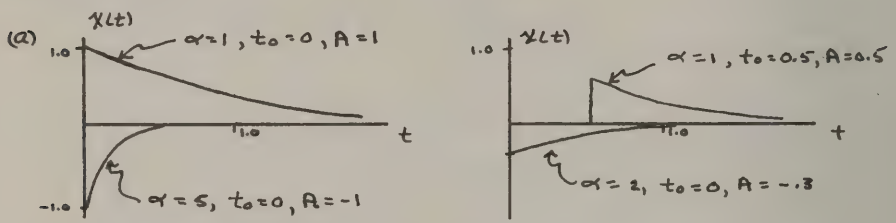
(a)



(b) $X(t)$ takes on only two values A and $-A$ and these are equally likely. Thus $f_X(x) = \frac{1}{2}\delta(x-A) + \frac{1}{2}\delta(x+A)$.

5.2

(a)

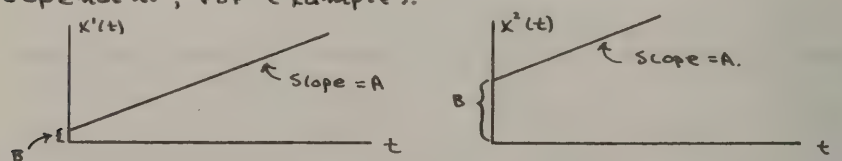


(b) This process is not stationary and thus not ergodic. The moments are not time-independent (e.g., $E\{X(t=t_0+\delta)\}$ = finite, $E\{X(t=\infty)\} = 0$.

5.3

(a) The process is stationary because averages do not depend on time origin. (Note that mean and variance are time-independent).

(b) Now $X(t)$ is non-stationary because statistical averages depend on time (the mean value is time-dependent, for example).



(5.4)

(a) $Y^i(t) = \int_0^t A \cos \omega\tau \, d\tau = \frac{A}{\omega} \sin \omega t$.

(b) $Y(t_0)$ is normal with mean zero and standard deviation $\sigma_A \frac{\sin \omega t_0}{\omega}$,

$$\therefore \quad f(y, t=t_0) = \frac{1}{\sqrt{2\pi \left(\frac{\sigma_A}{\omega} \sin \omega t_0\right)^2}} \, e^{-\frac{1}{2} y^2 / \left(\frac{\sigma_A}{\omega} \sin \omega t_0\right)^2}$$

(c) From part (b), $Y(t)$ is not stationary since standard deviation is time dependent. Also, since Ergodic $\Rightarrow$ Stationarity, it follows that Not stationary $\Rightarrow$ Not Ergodic. Thus $Y(t)$ is not Ergodic.

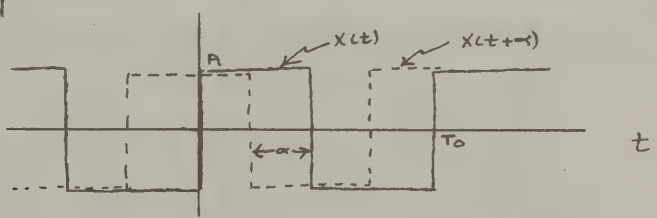
(5.5)

(a) By inspection the mean μ_X is zero.

$R(\alpha) = \langle X(t) \, X(t+\alpha) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) X(t+\alpha) \, dt$.

Note that since $X(t)$ is periodic, we need only integrate over a single period to find $R(\alpha)$ over that period. Then $R(\alpha)$ is also periodic.

Graphically:



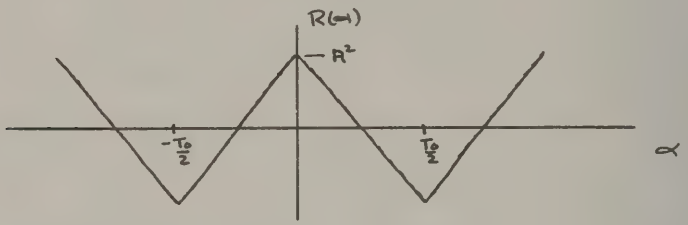
$$\therefore \quad R(\alpha) = \frac{1}{T_0} \left\{ A^2 \left(\frac{T_0}{2} - \alpha \right) - A^2 \alpha + A^2 \left(T_{0/2} - \alpha \right) - A^2 \alpha \right\}$$

$$= A^2 \left(1 - 4 \alpha / T_0\right) \quad \text{for} \quad 0 < \alpha < T_{0/2}.$$

(5.5) (Cont.)

$R(\alpha)$ is even and periodic so we can apply the properties of autocorrelation function to find $R(\alpha)$ for all α. The result is

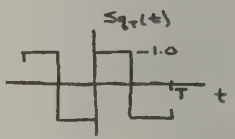
$$R(\alpha) = A^2\left(1 - 4\frac{|\alpha|}{T_0}\right) \quad |\alpha| \le T_0/2 \text{ , periodic in } T_0.$$



(b) By inspection $P\{X(t) = A\} = P\{X(t) = -A\} = 1/2$.

$\therefore E\{X(t)\} = \frac{1}{2}A - \frac{1}{2}A = 0.$

Define the function $Sq_T(t)$ as



Then $R_X(\tau) = E\{A\, Sq_{T_0}(t+\tau_0)\, A\, Sq_{T_0}(t+\tau_0+\tau)\}$

$$= A^2 \int_{-T_0/2}^{T_0/2} Sq_{T_0}(t+\tau_0)\, Sq_{T_0}(t+\tau+\tau_0)\, \frac{1}{T_0}\, d\tau_0.$$

Noting the similarity of this integral with that in part (a) we can write

$$R_X(\tau) = A^2\left(1 - 4\frac{|\tau|}{T_0}\right), \quad |\tau| \le \frac{T_0}{2}, \text{ periodic in } T_0.$$

(5.6)

$X(t) = A\cos(\omega_0 t + \Theta)$ where $f_\Theta(\theta) = \begin{cases} 4/\pi & , |\theta| \le \pi/8 \\ 0 & , \text{otherwise} \end{cases}$

(a) $E\{X(t)\} = \int_{-\pi/8}^{\pi/8} \frac{4}{\pi} A\cos(\omega_0 t+\theta)\,d\theta = \frac{4A}{\pi}\left[\sin(\omega_0 t+\theta)\right]\Big|_{-\pi/8}^{\pi/8}$

$= \frac{4A}{\pi}\left[\sin\omega_0 t \cos\frac{\pi}{8} + \cos\omega_0 t \sin\frac{\pi}{8} - \sin\omega_0 t \cos\frac{\pi}{8} + \cos\omega_0 t \sin\frac{\pi}{8}\right]$

$= \frac{8A}{\pi}\sin\frac{\pi}{8}\cos\omega_0 t$

$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t+\theta)\,dt = 0.$

$E\{X^2(t)\} = \int_{-\pi/8}^{\pi/8} \frac{4}{\pi} A^2\cos^2(\omega_0 t+\theta) = \frac{4A^2}{\pi}\int_{-\pi/8}^{\pi/8}\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t + 2\theta)\right]d\theta$

$= A^2/2 + \frac{A^2}{\pi}\left[\sin(2\omega_0 t + \pi/4) - \sin(2\omega_0 t - \pi/4)\right]$

$= A^2/2 + \frac{A^2}{\pi}\left[2\cos 2\omega_0 t \sin\pi/4\right]$

$\langle X^2(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos^2(\omega_0 t+\theta)\,dt = A^2/2.$

The variance is found using $VAR(X(t)) = E\{X^2(t)\} - E^2\{X(t)\}$.

(b) $R(\tau) = E\{X(t)X(t+\tau)\} = \int_{-\pi/8}^{\pi/8} \frac{4}{\pi} A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\,d\theta$

$= \frac{4A^2}{\pi}\int_{-\pi/8}^{\pi/8}\left[\frac{1}{2}\cos(2\omega_0 t + \omega_0\tau + 2\theta) + \frac{1}{2}\cos(\omega_0\tau)\right]d\theta$

$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{\pi}\left[\sin(2\omega_0 t + \omega_0\tau + \pi/4) - \sin(2\omega_0 t + \omega_0\tau - \pi/4)\right]$

$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{\pi}\left[2\cos(2\omega_0 t + \omega_0\tau)\sin\pi/4\right].$

Note that $R(\tau)$ is also time dependent.

$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\,dt$

$= \lim_{T\to\infty} \frac{A^2}{4T}\int_{-T}^{T}\left[\cos(2\omega_0 t + \omega_0\tau + 2\theta + \cos(\omega_0\tau)\right]dt$

$= \frac{A^2}{2}\cos(\omega_0\tau).$

This process is not stationary and thus is not ergodic.

(5.7)

Let X denote the random variable.

Then $Y = |X|$ is the rectified output.

$E\{Y^2\} = E\{|X|^2\} = E\{X^2\} = \sigma^2$.

$E\{|X|\} = 2\int_0^\infty x f_X(x)\,dx$ (by symmetry)

$= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^\infty x e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-u/2\sigma^2} du$.

The last expression follows by change of variables $u = x^2$. Thus

$E\{|X|\} = \frac{2\sigma^2}{\sqrt{2\pi\sigma^2}} = \sqrt{\frac{2}{\pi}}\,\sigma$.

$\text{Var}\{Y\} = E\{Y^2\} - E^2\{Y\} = \sigma^2 - \frac{2}{\pi}\sigma^2 = .363\sigma^2$

(5.8)

(a) $E\{Z(t)\} = E\{X_1 \cos\omega_0 t - X_2 \sin\omega_0 t\}$

$= E\{X_1\}\cos\omega_0 t - E\{X_2\}\sin\omega_0 t = 0$.

$E\{Z^2(t)\} = E\{[X_1\cos\omega_0 t - X_2 \sin\omega_0 t]^2\}$

$= E\{X_1^2\cos^2\omega_0 t - 2X_1X_2 \sin\omega_0 t \cos\omega_0 t + X_2^2 \sin^2\omega_0 t\}$

$= E\{X_1^2\}\cos^2\omega_0 t + E\{X_2^2\}\sin^2\omega_0 t - E\{X_1X_2\}\sin 2\omega_0 t$

$= \sigma^2 - E\{X_1\}E\{X_2\}\sin 2\omega_0 t = \sigma^2$.

$\text{VAR}\{Z(t)\} = \sigma^2$

(b) Z is Gaussian with zero mean, variance σ^2.

(5.9)

(a) $E\{Z^2\} = E\{[M(t)\cos\omega_0 t]^2\} = E\{M^2(t)\}\cos^2\omega_0 t$

$= M_0 \cos^2\omega_0 t$.

$Z(t)$ is not stationary since the 2nd moment is dependent on time.

5.9 (Cont.)

(b) $E\{z^2\} = E\{[m(t)\cos(\omega_0 t+\theta)]^2\} = E\{m^2(t)\cos^2(\omega_0 t+\theta)\}$

$= E\{m^2(t)\}\, E\{\cos^2(\omega_0 t+\theta)\}$ (by independence)

$= M_0\, E\{\tfrac{1}{2} + \cos(\omega_0 t+\theta)\} = \frac{M_0}{2} + M_0\, E\{\cos(\omega_0 t+\theta)\}$

$= M_0/2.$

Can't say if $z(t)$ is stationary from the information we have here, but it turns out that it is.

5.10

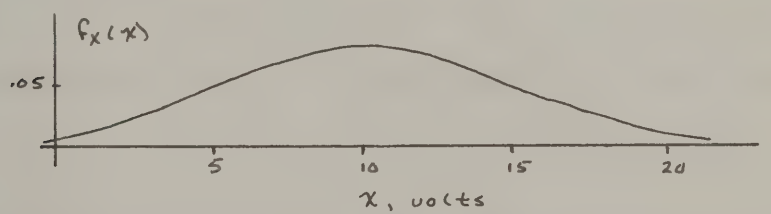
Let $x(t)$ be the output voltage.

The DC meter reading gives $E\{x(t)\} = 10.$

The true RMS meter reading gives $[E\{(X - E\{X\})^2\}]^{1/2}$ which is the standard deviation.

$\therefore$ X is approximately Gaussian with mean 10 and variance 25.

$f_X(x) = \frac{1}{\sqrt{50\pi}}\, e^{-\frac{1}{50}(x-10)^2}.$



5.11

(a) Not suitable, $R(\tau) \neq R(-\tau)$

(c) Not suitable, $\mathcal{F}\{R(\tau)\} < 0$ for some f.

(b), (d) Suitable, both are symmetric, $\mathcal{F}\{R(\tau)\} \geq 0\ \forall f$, maximum occurs at $\tau = 0$, etc.

5.12

(a) $P_o = BN_o$. For this case $B = 1\,KHz.$, $\frac{N_o}{2} = 2\times10^{-6}\ v^2/hz.$

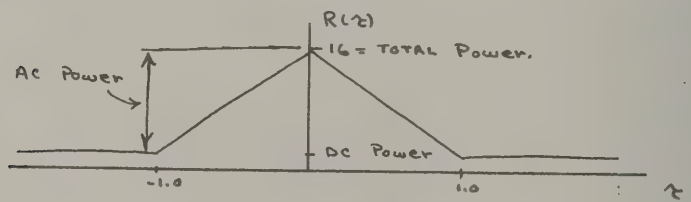
$\therefore P_o = 4\times10^{-3}\ v^2.$

(b) The amplitude pdf is Gaussian, zero mean, and process is stationary. $\sigma_N^2 = P_o = 4\times10^{-3}$ because of zero mean.

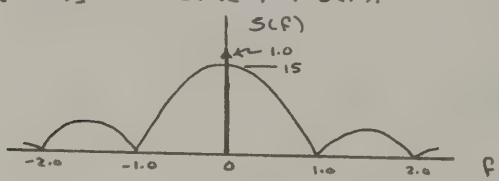
$f_X(x) = \frac{1}{\sqrt{8\times10^{-3}\pi}}\, e^{-\frac{1}{8\times10^{-3}}x^2}$.

5.13

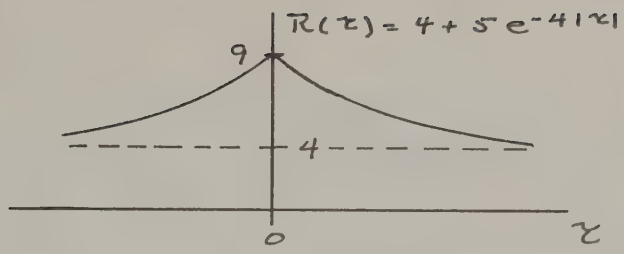
(a) $R(0) = RMS^2 = 16$, $R(\pm\infty) = DC = 1$.



(b) $S(f) = \mathcal{F}\{R(\tau)\} = 15\,\mathrm{sinc}^2 f + \delta(f)$.



5.14



As $|\tau| \to \infty$, $R(\tau) \to (\text{mean})^2$

(a) ac power = $R(0) - (\text{mean})^2 =$
$9 - 4 = 5$ Watts

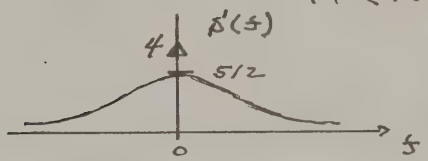
(b) dc power = $(\text{mean})^2 = 4$ Watts

(c) Total power = $R(0) = 9$ Watts

(d) $S(f) = \mathcal{F}\{R(\tau)\} = \mathcal{F}\{4 + 5e^{-4|\tau|}\}$

$$= 4\delta(f) + \frac{2\left(\frac{1}{4}\right)(5)}{1 + (2\pi f/4)^2}$$

$$= 4\delta(f) + \frac{5}{2}\,\frac{1}{1 + (\pi f/2)^2}$$



5.15

$$R_Y(\tau) = E\{Y(t)\,Y(t+\tau)\}$$
$$= E\{[X(t) + X(t-T)][X(t+\tau) + X(t+\tau-T)]\}$$
$$= E\{X(t)X(t+\tau)\} + E\{X(t)X(t+\tau-T)\}$$
$$+ E\{X(t-T)X(t+\tau)\} + E\{X(t-T)X(t+\tau-T)\}$$
$$= R_X(\tau) + R_X(\tau-T) + R_X(\tau+T) + R_X(\tau)$$
$$= 2R_X(\tau) + R_X(\tau-T) + R_X(\tau+T)$$

$$S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = 2S_X(f) + S_X(f)\left[e^{-j2\pi fT} + e^{j2\pi fT}\right]$$
$$= 2S_X(f) + 2S_X(f)\cos 2\pi fT$$
$$= 2S_X(f)\left[1 + \cos 2\pi fT\right]$$
$$= 4S_X(f)\cos^2 \pi fT$$

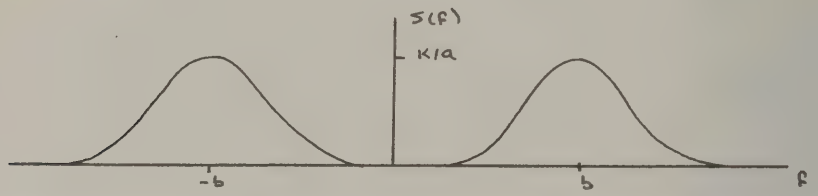
(5.16)

$S(f) = \mathcal{F}\{R(\tau)\} = \mathcal{F}\{K e^{-a|\tau|} \cos 2\pi b\tau\}$,

$\mathcal{F}\{K e^{-a|\tau|}\} = \dfrac{2K/a}{1 + (2\pi f/a)^2}$.

Applying the modulation theorem gives

$$S(f) = \frac{K}{a}\left[\frac{1}{1 + \frac{4\pi^2}{a^2}(f-b)^2} + \frac{1}{1 + \frac{4\pi^2}{a^2}(f+b)^2}\right].$$



(5.17)

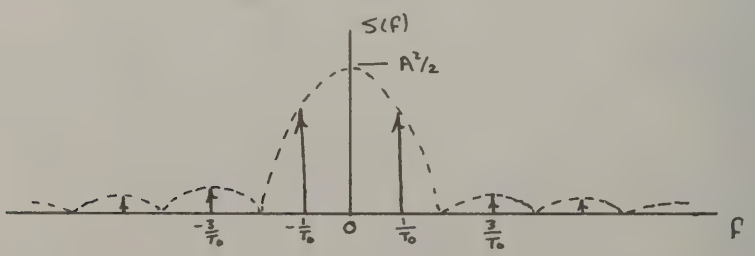
From prob. (5.1) we have R(τ) a triangular waveform with zero DC value. The magnitude is A^2 and period is T_0. Since R(τ) is periodic, we can express it as a Fourier Series. Thus

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n e^{j 2\pi n \tau/T_0} \quad \text{where}$$

$$C_n = \begin{cases} A^2/2 \, \text{sinc}^2 n/2 & \text{for } n \neq 0 \\ 0 & \text{for } n = 0 \end{cases} = \begin{cases} \left(\frac{2}{\pi n}\right)^2 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Fourier transforming gives

$$S(f) = \mathcal{F}\{R(\tau)\} = \sum_{n=-\infty}^{\infty} C_n \, \delta(f - n/T_0).$$



(5.18)

The random process is $X(t) = n(t) \sum_{K=-\infty}^{\infty} \delta(t - KT_s)$,

where $E\{n(t)\, n(t+\tau)\} = R_n(\tau)$.

$$N_T^{(i)}(f) = \int_{-T/2}^{T/2} n^{(i)}(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} n^{(i)}(t) \sum_{K=-\infty}^{\infty} \delta(t - KT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{K=-N}^{N} n^{(i)}(KT_s)\, e^{-j\omega KT_s}$$

where N = largest integer such that $NT_s \le T/2$.

$$|N_T^{(i)}(f)|^2 = \left| \left[\sum_{K=-N}^{N} n^{(i)}(KT_s)\, e^{-j\omega KT_s} \right] \right|^2.$$

Now we need to find $E\{|N_T^{(i)}(f)|^2\}$. Note that the result will be a sum of terms of the form

$E\{[n^{(i)}(KT_s)\, e^{-j\omega KT_s}][n^{(i)}(MT_s)\, e^{-j\omega MT_s}]^*\}$.

But since $R_N(KT_s) = 0$ for $K = 1, 2, \ldots$ the result is

$$E\{|N_T^{(i)}(f)|^2\} = \sum_{K=-N}^{N} E\{[n^{(i)}(KT_s)\, e^{-j\omega KT_s}][n^{(i)}(KT_s)\, e^{-j\omega KT_s}]^*\}$$

$$= \sum_{K=-N}^{N} R_N(0) = 2N R_n(0).$$

From the text: $S_N(f) = \lim_{T\to\infty} \frac{1}{T} E\{|N_T^{(i)}(f)|^2\}$.

Thus $S_X(f) = \lim_{T\to\infty} \frac{1}{T} \sum_{K=-N}^{N} R_N(0) = \lim_{T\to\infty} \frac{2N}{T} R_N(0)$.

We can replace T with $2NT_s$ and let $N \to \infty$.

This gives

$$S_X(f) = \lim_{N\to\infty} \frac{2N R_n(0)}{2N T_s} = \frac{1}{T_s} R_n(0).$$

(5.19)

$X(t) = \sum_{K=-\infty}^{\infty} a_K \delta(t - t_K)$ where a_K, t_K are independent, and $E\{a_K a_L\} = \sigma_a^2 \delta_{KL}$, $P_R\{N\ t_K\text{'s in } T \text{ secs.}\} = \frac{(\nu T)^N}{N!} e^{-\nu T}$

(a) $X_T^{(i)} = X^{(i)}(t)$, $|t| < T/2$,

$$= \left[\sum_{K=-\infty}^{\infty} a_K \delta(t - t_K) \right] [u(t + T/2) - u(t - T/2)].$$

$$N_T^{(i)}(f) = \int_{-T/2}^{T/2} \sum_{K=-\infty}^{\infty} a_K \delta(t - t_K)\, e^{-j\omega t}\, dt = \sum_{\text{ALL } K \ni |t_K| < T/2} a_K e^{-j\omega t_K}.$$

(5.19) (Cont.)

$$|N_T^{(i)}(f)|^2 = \sum_{\text{All } K \ni |t_K| < T/2} (a_K e^{-j\omega t_K})(a_K e^{-j\omega t_K})^* + \sum_{K \neq L} a_K a_L \text{ type terms}$$

$$E\{|N_T^{(i)}(f)|^2\} = E\left\{ \sum_{\text{All } K \ni |t_K| < T/2} a_K a_K^* \right\} = E\{a_K a_K^*\} E\left\{ \sum_{\text{All } K \ni |t_K| < T/2} 1 \right\},$$

$$= \sigma_a^2 \nu T.$$

$$\therefore S_x(f) = \lim_{T\to\infty} \frac{1}{T} |N_T^{(i)}(f)|^2 = \lim_{T\to\infty} \frac{\sigma_a^2 \nu T}{T} = \sigma_a^2 \nu;$$

(b) $x(t)$, $S_x(f)$ → $h(t)$, $H(f)$ → $y(t)$, $S_y(f)$

$$S_y(f) = |H(f)|^2 S_x(f) = \sigma_a^2 \nu |H(f)|^2.$$

(c) $$R_y(\tau) = \mathcal{F}^{-1}\{S_y(f)\} = \int_{-\infty}^{\infty} |H(f)|^2 S_x(f) e^{j2\pi f} df$$

which reduces to

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau) = \sigma_a^2 \nu\, h(\tau) * h(-\tau).$$

(d) Recall that N represents the number of t_K's in an interval T. The characteristic function of $y(t)$ is $M_y(jv) = E\{e^{jvy(t)}\} = E\{e^{jv \sum_N a_K h(t-t_K)}\}$, where N, a_K, and t_K are all random variables. We can first look at the conditional C.F. $M_y(jv/N)$ and later find $M_y(jv)$ using

$$M_y(jv) = \sum_{N=0}^{\infty} M_y(jv/N) P_r(n).$$

Now $$M_y(jv/N) = E\{e^{jv \sum_N a_K h(t-t_K)}\}$$

$$= E\left\{ \prod_N e^{jv a_K h(t-t_K)} \right\} = \prod_N E\{e^{jv a_K h(t-t_K)}\}$$

which follows by independence

$$= [E\{e^{jv a_K h(t-t_K)}\}]^N$$

which follows by identically distributed

(5.19) (Cont.)

$$= \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_A(a)\, e^{jva\,h(t)}\, da\, dt \right]^N .$$

which follows by change of variables.

$$\therefore M_y(jv) = \sum_{N=0}^{\infty} \left[\iint_{-\infty}^{\infty} f_A(a)\, e^{jva\,h(t)}\, da\, dt \right]^N \frac{(vT)^N}{N!} e^{-vT}$$

$$= \exp\left[vT \iint_{-\infty}^{\infty} f_A(a)\, e^{jva\,h(t)}\, da\, dt \right] e^{-vt}$$

$$= \exp\left[vT \left\{ \iint_{-\infty}^{\infty} f_A(a)\, e^{jva\,h(t)}\, da\, dt - 1 \right\} \right] .$$

(c) Differentiate $M_y(jv)$ to find $E\{y(t)\}$ and $E\{y^2(t)\}$.

(5.20)

Denote the waveform $x(t)$. Then form the truncated version $x_{2nT}(t)$ given by

$$x_{2nT}(t) = \begin{cases} x(t) & , |t| \le nT \\ 0 & , \text{otherwise} \end{cases}$$

Note that there are $2n$ pulses in $x_{2nT}(t)$. The Fourier transform of the k^{th} pulse $x_k(t)$ is

$$X_k(f) = \int_{-T}^{T} x_k(t)\, e^{-j2\pi ft}\, dt = \int_{T+\Delta t_k}^{T+\tau_0+\Delta t_k} A e^{-j2\pi ft}\, dt$$

$$= A\tau_0 \operatorname{sinc} f\tau_0 \exp[-j2\pi f(\Delta t_k + \tau_0/2 + kT)]$$

It follows then that

$$F\{x_{2nT}(t)\} = \sum_{k=-n}^{n-1} A\tau_0 \operatorname{sinc} \tau_0 f \exp[-j2\pi f(\Delta t_k + \tau_0/2 + kT)]$$

$$= A\tau_0 \operatorname{sinc} \tau_0 f\, e^{-j2\pi f\tau_0/2} \sum_{k=-n}^{n-1} e^{-j2\pi f(\Delta t_k + kT)}$$

and so

$$|X_{2nT}(f)|^2 = A^2\tau_0^2 \operatorname{sinc}^2 \tau_0 f \sum_{k=-n}^{n-1} \sum_{l=-n}^{n-1} e^{-j2\pi f(\Delta t_k + kT)}\, e^{-j2\pi f(\Delta t_l + lT)}$$

Taking the expected value gives

$$E\{|X_{2nT}(f)|^2\} = A\tau_0^2 \operatorname{sinc}^2 \tau_0 f \sum_{k=-n}^{n-1} \sum_{l=-n}^{n-1} E\left\{e^{-j2\pi f(\Delta t_k + kT)} + e^{+j2\pi f(\Delta t_l + lT)}\right\}$$

Note that for $k = l$, the expectation inside summation is 1. Now suppose $k \neq l$. Then

$$E\{e^{-j2\pi f(\Delta t_k + kT)}\, e^{+j2\pi f(\Delta t_k + lT)}\}$$

$$= e^{-j2\pi f(k-l)T}\, E\{e^{-j2\pi f\Delta t_k}\}\, E\{e^{-j2\pi f\Delta t_l}\}.$$

With

$$E\{e^{-j2\pi f\Delta t_k}\} = \frac{4}{\tau} \int_0^{\tau/4} e^{-j2\pi f\Delta t_k}\, d\Delta t_k = \operatorname{sinc}\left(\frac{fT}{4}\right) e^{-j\pi fT/4}$$

the mean squared expression becomes

$$E\{|X_{2nT}(f)|^2\} = A^2\tau_0^2 \operatorname{sinc}^2 \tau_0 f \left\{ \sum_{\substack{k=-n \\ k\neq l}}^{n-1} \sum_{l=-n}^{n-1} e^{-j2\pi f(k-l)T} \operatorname{sinc}\left(\frac{fT}{4}\right) + \sum_{k=-n}^{n-1} 1 \right\}$$

$$= 2nA^2\tau_0^2 \operatorname{sinc}^2 \tau_0 f \left[1 - \operatorname{sinc}\left(\frac{fT}{4}\right)\right] + A^2\tau_0^2 \operatorname{sinc}^2 \tau_0 f \underbrace{\sum_{k=-n}^{n-1} e^{-j2\pi fkT} \sum_{l=-n}^{n-1} e^{j2\pi flT}}_{\triangleq\ |S_n|^2}$$

(5.20) (Cont.)

where the last expression results from including $k=l$ in summation. Next we need to evaluate S_n.

$$S_n = \sum_{k=-n}^{n-1} e^{-j2\pi f_k T} = \sum_{k=0}^{n-1} e^{-j2\pi f_k T} + \sum_{k=-n}^{-1} e^{-j2\pi f_k T}$$

$$= (1+e^{j2\pi f}) \sum_{k=1}^{n} e^{-j2\pi f_k T}.$$

Thus

$$S_n = (1+e^{j2\pi fT}) \frac{e^{-j2\pi fT}\left[1-e^{-j2\pi nfT}\right]}{1-e^{-j2\pi fT}}$$

where $\sum_{i=1}^{n} z^n = \frac{z(1-z^n)}{1-z}$ has been used.

This expression simplifies to

$$S_n = e^{-j\pi nfT} \frac{2\cos\pi fT \sin n\pi fT}{\sin \pi fT}$$

and so

$$|S_n|^2 = \frac{4\cos^2\pi fT \sin^2 n\pi fT}{\sin^2 \pi fT}$$

Therefore, the power spectral density of $x(t)$ is written

$$S_x(f) = \lim_{n\to\infty} \frac{E\{|X_{2nT}(f)|^2\}}{2nT}$$

$$= \lim_{n\to\infty}\left[\frac{1}{2nT} A^2\tau_0^2 \sin c^2 \tau_0 f\left\{2n(1-\sin cfT/4) + \frac{4\cos^2\pi fT \sin^2 n\pi fT}{\sin^2 \pi fT}\right\}\right]$$

$$= \frac{A^2\tau_0^2}{T}\sin c^2\tau_0 f\left(1-\sin c\frac{fT}{4}\right) + \frac{2A^2\tau_0^2}{T}\sin c^2\tau_0 f \cos^2\pi fT \left[\lim_{n\to\infty}\frac{\sin^2 n\pi fT}{n\sin^2\pi fT}\right]$$

The limit in the last term of this expression is examined next.

(5.20) (Cont.)

The limit

$$L = \lim_{n\to\infty} \left[\frac{\sin^2 n\pi fT}{n \sin^2 \pi fT}\right]$$

is zero for $f \neq m/T$, m an integer because the numerator is finite and denominator goes to infinity. Now let f be ϵ away from m/T. Then

$$L = \lim_{n\to\infty} \frac{\sin^2 [2n(m\pi + \pi\epsilon)]}{2n[\pi\epsilon(-1)^m]^2}$$

$$= \lim_{n\to\infty} \frac{\sin^2(2n\pi\epsilon)}{2n\pi^2\epsilon^2}.$$

Now it was shown in Chapter 2 that

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \operatorname{sinc}^2 \epsilon x \quad (x \text{ near } 0)$$

$$= \lim_{n\to\infty} n \operatorname{sinc}^2 nx$$

and similarly it can be shown that

$$L = \lim_{n\to\infty} \frac{\sin^2(2n\pi\epsilon)}{2n\pi^2\epsilon^2} = \delta(f - m/T).$$

Thus

$$S_x(f) = \frac{A^2\tau_0^2}{T} \operatorname{sinc}^2 \tau_0 f \left[(1 - \operatorname{sinc}\tfrac{fT}{4}) + 2\delta(f - m/T)\right].$$

(5.21)

(a) The filter causes an attenuation γ of $\gamma = [1 + \omega_0^2 R^2 C^2]^{-1/2}$, and introduces an additional phase angle of $\psi = -\tan^{-1} \omega_0 RC$.

$\therefore \quad B = A[1+\omega_0^2 R^2 C^2]^{-1/2}, \quad \phi = \theta - \tan^{-1} \omega_0 RC.$

(b) $R_{xy}(\tau) = E\{x(t)y(t+\tau)\}$

$= E\{A \sin(\omega_0 t + \theta) B \sin(\omega_0 t + \omega_0 \tau + \phi)\}$

$= \frac{AB}{2} E\{\cos(\theta - \omega_0 \tau - \phi) - \cos(2\omega_0 t + \omega_0 \tau + \theta + \phi)\}$

$= \frac{AB}{2} \cos(\tan^{-1} \omega_0 RC - \omega_0 \tau)$

$= \frac{A^2}{2} [1+\omega_0^2 R^2 C^2]^{-1/2} \cos(\tan^{-1} \omega_0 RC - \omega_0 \tau).$

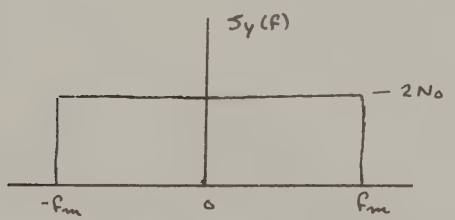
(c) $R_{yx}(\tau) = R_{xy}(-\tau) = \frac{A^2}{2} [1+\omega_0^2 R^2 C^2] \cos(\tan^{-1} \omega_0 RC + \omega_0 \tau).$

(5.22)

Let $S_y(f)$ denote output power spectral density.

Then $S_y(f) = S_m(f) |H(f)|^2.$

$$= \begin{cases} 2N_0 & |f| < f_m \\ 0 & \text{elsewhere.} \end{cases}$$



(5.23)

Let $S_x(f)$ denote output P.S.D. and $H(f)$ the filter response.

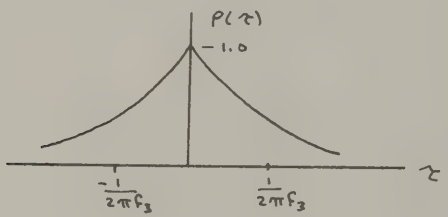
(a) $H(f) = \dfrac{1}{1+jf/f_3}$.

$S_x(f) = S_m(f)\,|H(f)|^2 = \dfrac{N_0}{2}\,\dfrac{1}{1+(f/f_3)^2}$.

$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \displaystyle\int_{-\infty}^{\infty} \frac{N_0}{2}\,\frac{1}{1+(f/f_3)^2}\,e^{-j2\pi f\tau}\,d\tau$

$= \dfrac{N_0}{2}\pi f_3\, e^{-2\pi f_3|\tau|}$.

(b)



(c) $\rho(\tau) = e^{-2\pi f_3|\tau|} \le 0.1$

$\ln(0.1) = -2.303$, so $\quad 2\pi f_3|\tau| \ge 2.303$

and so $\quad f_3|\tau| \ge .366$

(5.24) Except for part (d), this problem is identical to Example 5.9

(a) $H(f) = \dfrac{1/j2\pi RC}{1 + 1/j2\pi RC} = \dfrac{1}{1 + j2\pi RC}$

$$S_o(f) = S_i'(f) = \frac{1}{2} N_0 \frac{1}{1 + (2\pi RC)^2}$$

(b) Since $\exp(-|\tau|/\alpha) \longleftrightarrow \dfrac{2\alpha}{1 + (2\pi f\alpha)^2}$

For this case $\alpha = RC$. Thus

$$R_o(\tau) = \frac{N_0}{4RC} \exp(-|\tau|/RC)$$

(c) The output is zero-mean, Gaussian and has variance $R_o(0) = \dfrac{N_0}{4RC}$

$$S_y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2/2\sigma_y^2} = \frac{1}{\sqrt{\pi N_0/2RC}} e^{-y^2/(N_0/2RC)}$$

(d) The output noise power $= N_0 B_N$. From part (b), the output noise power $= \dfrac{N_0}{4RC}$ Thus $N_0 B_N = \dfrac{N_0}{4RC}$ and $B_N = 1/4RC$

(5.25)

$H(f)$ = filter response $= \dfrac{1}{1 + j2\pi fRC}$

(a) $R_n(\tau) = \mathcal{F}^{-1}\{S_n'(f)\} = \mathcal{F}^{-1}\left\{\dfrac{N_0}{2}\right\} = \dfrac{N_0}{2}\delta(\tau)$

(b) $S_y'(f)$ = output Power Spectral Density

$$= S_n'(f)|H(f)|^2 \quad (4)$$

$$= 2N_0 \frac{1}{1 + (2\pi fRC)^2}$$

(c) $R_y(\tau) = \mathcal{F}^{-1}\{S_y'(f)\} = \dfrac{2N_0}{2RC} e^{-|\tau|/RC}$

$$= \frac{N_0}{RC} e^{-|\tau|/RC}$$

(5.26)

(a) $n(t) = \sum_{i=1}^{N} n_i(t)$.

$E\{n(t)\} = 0$ since $E\{n_i(t)\} = 0 \quad \forall i$.

$E\{n^2(t)\} = E\{[\sum_{i=1}^{N} n_i(t)][\sum_{j=1}^{N} n_j(t)]\} = E\{\sum_{i=1}^{N}\sum_{j=1}^{N} n_i(t)\, n_j(t)\}$

$= \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} E\{n_i(t)\, n_j(t)\}}$

N terms of the form $n_i(t)\, n_i(t) \Rightarrow \sigma^2$

$2N+2$ terms of the form $n_i(t)\, n_{i\pm1}(t) \Rightarrow \sigma^2/2$.

$\therefore E\{n^2(t)\} = N\sigma^2 + (2N+2)\sigma^2/2 = (2N+1)\sigma^2$.

$\mathrm{Var}\{n(t)\} = E\{n^2(t)\}$ since zero mean.

(b) $f_N(n) = \dfrac{1}{\sqrt{2\pi(2N+1)\sigma^2}}\, e^{-\frac{1}{2} n^2/(2N+1)\sigma^2}$

(5.27)

Let $R(\tau) = E\{r(t)\, r(t+\tau)\}$.

(a) $R(\tau) = E\{r(t)\, r(t+\tau)\} = E\{n(t)\cos(\omega_0 t+\phi)\, n(t+\tau)\cos(\omega_0 t+\omega_0\tau+\phi)\}$

$= E\{n(t)\, n(t+\tau)\}\, E\{\cos(\omega_0 t+\phi)\cos(\omega_0 t+\omega_0\tau+\phi)\}$

$= R_n(\tau)\, E\{\frac{1}{2}\cos\omega_0\tau + \frac{1}{2}\cos(2\omega_0 t+\omega_0\tau+2\phi)\}$

$= \frac{1}{2} R_n(\tau)\cos\omega_0\tau$.

(B) By the modulation theorem,

$S(f) = \frac{1}{4} S_n(f+f_0) + \frac{1}{4} S_n(f-f_0)$.

(5.28)

(a) $y(t) = \frac{1}{T}\int_t^{t+T} x(\lambda)\, x(\lambda+\tau)\, d\lambda$

$E\{y(t)\} = E\{\frac{1}{T}\int_t^{t+T} x(\lambda)\, x(\lambda+\tau)\, d\lambda\} = \frac{1}{T}\int_t^{t+T} E\{x(\lambda)\, x(\lambda+\tau)\}\, d\lambda$

$= \frac{1}{T}\int_t^{t+T} R_x(\tau)\, d\lambda = R_x(\tau)\, \frac{1}{T}\int_t^{t+T} d\lambda = R_x(\tau)$.

(5.28) (Cont.)

(b) $E\{y^2(t)\} = E\left\{\left[\frac{1}{T}\int_t^{t+T} x(\lambda)\,x(\lambda+\tau)\,d\lambda\right]\left[\frac{1}{T}\int_t^{t+T} x(\beta)\,x(\beta+\tau)\,d\beta\right]\right\}$

$$= \frac{1}{T^2}\int_t^{t+T}\int_t^{t+T} \underbrace{E\{x(\lambda)\,x(\lambda+\tau)\,x(\beta)\,x(\beta+\tau)\}}\,d\lambda\,d\beta$$

$$= R_x^2(\tau) + R_x^2(\lambda-\beta) + R_x(\lambda-\beta-\tau)R_x(\lambda+\tau-\beta)$$

$$\therefore E\{y^2(t)\} = \frac{1}{T^2}\int_t^{t+T}\int_t^{t+T}\left[R_x^2(\tau) + R_x^2(\lambda-\beta) + R_x(\lambda-\beta-\tau)R_x(\lambda+\tau-\beta)\right]d\lambda\,d\beta$$

and $\quad Var\{y(t)\} = E\{y^2(t)\} - R_x^2(\tau).$

(5.29)

$S_n(f) = \frac{N_0}{2}$, $\quad H(f) = \begin{cases} 1, & |f|<B \\ 0, & \text{elsewhere} \end{cases}$, $\quad S_x(f)$ = P.S.D. of output.

(a) $S_x(f) = S_n(f)\,|H(f)|^2 = \begin{cases} N_0/2, & |f|<B \\ 0, & \text{elsewhere.} \end{cases}$

$$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f\tau}\,df = N_0 B\,\text{sinc}\,2B\tau.$$

(b) $E\{x(t)\} = 0$, $\quad R_x(\tfrac{1}{2B}) = 0$, $\quad Var(x(t)) = R_x(0) = N_0B.$

Note uncorrelated Gaussian $\Rightarrow$ independent.

$$\therefore f(x_1, x_2) = \frac{1}{2\pi N_0 B}\, e^{-(x_1^2+x_2^2)/2N_0B}.$$

(5.30)

(a) $R_{y\dot{y}}(\tau) = E\left\{y(t)\,\frac{\partial y(t+\tau)}{\partial t}\right\} = E\left\{y(t)\left[\lim_{\varepsilon\to 0}\frac{y(t+\tau+\varepsilon)-y(t+\tau)}{\varepsilon}\right]\right\}$

$= E\left\{\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[y(t)\,y(t+\tau+\varepsilon)-y(t)\,y(t+\tau)\right]\right\}$

$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E\{y(t)\,y(t+\tau+\varepsilon)\}-E\{y(t)\,y(t+\tau)\}\right]$

$= \lim_{\varepsilon\to 0}\frac{R_y(\tau+\varepsilon)-R_y(\tau)}{\varepsilon} = \frac{\partial R_y(\tau)}{\partial \tau}$

(b) $R_y(\tau) = \mathcal{F}^{-1}\left\{\frac{N_0}{2}\,\Pi\left(\frac{f}{2B}\right)\right\} = N_0 B\,\mathrm{sinc}\,2B\tau$

Since $R_y(\tau)$ is even and continuous at $\tau = 0$, it follows that

$$\left.\frac{\partial R_y(\tau)}{\partial \tau}\right|_{\tau=0} = 0$$

Since $Y \triangleq y(t)$ and $Z \triangleq \frac{\partial}{\partial t}y(t)$ are Gaussian and uncorrelated, independent.

$$\sigma_y^2 = \tfrac{1}{2}N_0\int_{-\infty}^{\infty}\Pi\left(\frac{f}{2B}\right)df = N_0 B$$

$$\sigma_{\dot{y}}^2 = \tfrac{1}{2}N_0\int_{-\infty}^{\infty}(2\pi f)^2\,\Pi\left(\frac{f}{2B}\right)df = 2N_0\pi^2\int_{-B}^{B}f^2\,df = \frac{4\pi^2 N_0 B^3}{3}$$

Thus

$$f_{y\dot{y}}(y,\dot{y}) = \frac{1}{2\pi\sqrt{\frac{4\pi^2 N_0^2 B^4}{3}}}\,e^{-y/2N_0B}\,e^{-\dot{y}/8\pi^2 N_0 B^2/3}$$

$$= \frac{\sqrt{3}}{4\pi^2 N_0 B^2}\exp\left\{-\left(\frac{y}{2N_0B}+\frac{3\dot{y}}{8\pi^2 N_0 B^2}\right)\right\}$$

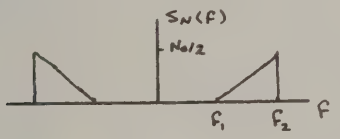
(c) N_0. For that case the PSD at the output is $\frac{(2\pi f)^2}{1+(f/f_3)^2}$ where $f_3 = \frac{1}{2\pi RC}$. Note that this does not go to zero as $f\to\infty$. The output power is therefore infinite.

(5.31)

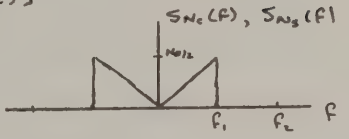
From Eqn. 5-41 we have

$$S_{N_c}(f) = S_{N_s}(f) = LP[S_N(f-f_c) + S_N(f+f_c)]$$

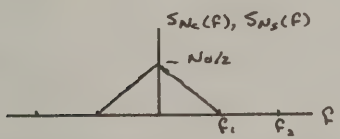
$$S_{N_c N_s}(f) = j\,LP[S_N(f-f_c) - S_N(f+f_c)]$$



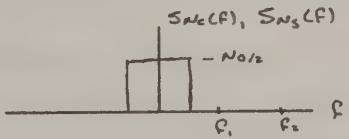
(a)



(b)



(c)



(d) $N_c(t)$, $N_s(t)$ are uncorrelated if $E\{N_c(t)N_s(t)\} = 0$.

$$E\{N_c(t)N_s(t)\} = R_{N_c N_s}(\tau) = \int_{-\infty}^{\infty} S_{N_c N_s}(f)\,df.$$

Thus $\int_{-\infty}^{\infty} S_{N_c N_s}(f)\,df$ must be zero in order for $n_c(t)$ and $n_s(t)$ to be uncorrelated. This is not the case for any of the three examples.

5.32

(a) $R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\} = \int_{-\infty}^{\infty} \frac{\alpha^2}{\alpha^2+(2\pi f)^2} e^{j2\pi f\tau}\, df = \frac{\alpha}{2} e^{-\alpha|\tau|}$

$\therefore K = \frac{\alpha}{2}$.

(b) Using the result of part (a) and modulation theorem

$R_N(\tau) = \frac{\alpha}{2} e^{-\alpha|\tau|} \cos \omega_c \tau$.

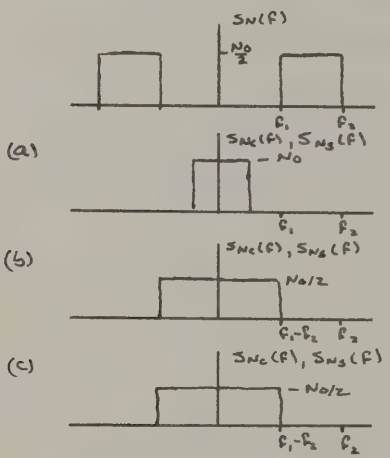
(c) $S_{N_c}(f) = S_{N_s}(f) = LP[S_N(f-f_c) + S_N(f+f_c)]$

$S_{N_c N_s}(f) = j\, LP[S_N(f-f_c) - S_N(f+f_c)]$.

Thus $S_{N_c}(f) = S_{N_s}(f) = \frac{\alpha^2}{\alpha^2+\omega^2}$.

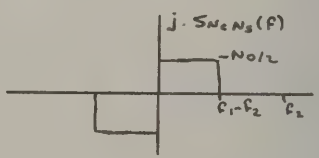
$S_{N_c N_s}(f) = 0$ since $S_N(f)$ is symmetric about ω_c.

5.33



(a) $S_{N_c N_s}(f) = 0$

(b)



(c) $S_{N_c N_s}(f)$ is -1 times $S_{N_c N_s}(f)$ of part (b).

(d) For parts (b) and (c), $n_c(t)$ and $n_s(t)$ are correlated.

For part (c): $R_{N_c N_s}(\tau) = \mathcal{F}^{-1}\{S_{N_c N_s}(f)\}$

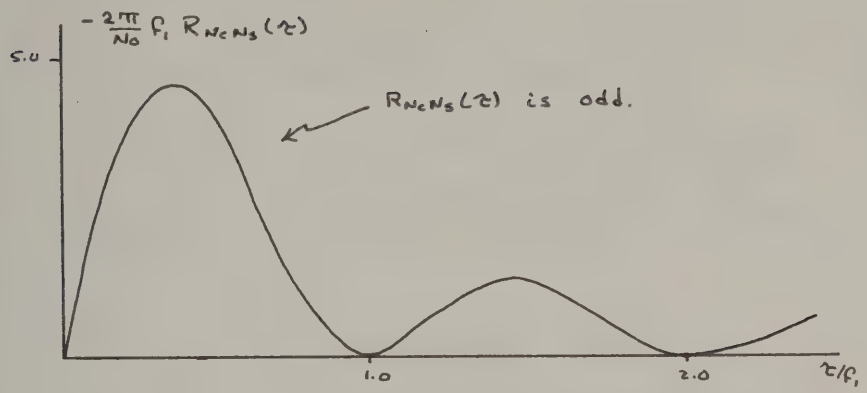
$= -j \int_{-f_1}^{0} \frac{N_0}{2} e^{j2\pi f\tau} df + j \int_{0}^{f_1} \frac{N_0}{2} e^{j2\pi f\tau} df$

$= -\frac{N_0}{2\pi\tau} + \frac{N_0}{4\pi\tau}\left[e^{-j2\pi f_1\tau} + e^{j2\pi f_1\tau}\right]$

$= \frac{N_0}{2\pi\tau}\left[\cos 2\pi f_1\tau - 1\right]$.

Similarly for part (b): $R_{N_c N_s}(\tau) = \frac{N_0}{2\pi\tau}\left[1 - \cos 2\pi f_1\tau\right]$.

(5.33) (Cont.)



(5.34)

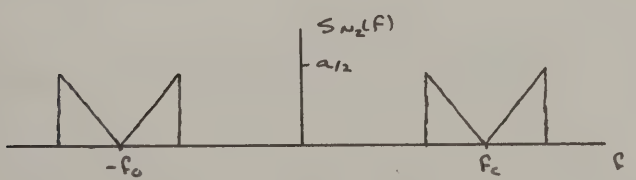
$$E\{n_2(t_1)n_2(t_2)\} = E\{n_1(t_1)[\cos(\omega_c t_1+\theta) - \sin(\omega_c t_1+\theta)]\, n_1(t_2)$$
$$[\cos(\omega_c t_2+\theta) - \sin(\omega_c t_2+\theta)]$$

By applying trig. identities, this reduces to

$$E\{n_2(t_1)n_2(t_2)\} = R_n(\tau)E\{\cos\omega_c\tau - \sin(\omega_c t_1+\omega_c t_2+2\theta)\}$$
$$= R_n(\tau)\cos\omega_c\tau.$$

Thus, by the modulation theorem

$$S_{N_2}(f) = \tfrac{1}{2}S_N(f-f_c) + \tfrac{1}{2}S_N(f+f_c).$$



(5.35)

Define $N_1 = A + N_c$.

Then $R = [(A+N_c)^2 + N_s^2]^{1/2} = [N_1^2 + N_s^2]^{1/2}$.

Note that the joint pdf of N_1, N_s is

$$f_{N_1,N_s}(n_1,n_s) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(n_1-A)^2 + n_s^2]}.$$

Thus

$$F_R(r) = Pr\{R \le r\} = Pr\{\sqrt{N_1^2+N_s^2} \le r\}$$

$$= \iint_{\sqrt{N_1^2+N_s^2} \le r} f_{N_1,N_2}(n_1,n_s)\, dn_1\, dn_s$$

$$= \iint_{\sqrt{N_1^2+N_s^2} \le r} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(n_1-A)^2+n_s^2]}\, dn_1\, dn_s$$

Now use change of variables from rectangular to polar coordinates. That is,

$$n_1 = \alpha\cos\beta \quad , \quad \alpha = [n_1^2 + n_s^2]^{1/2}$$

$$n_s = \alpha\sin\beta \qquad \beta = \tan^{-1}(n_s/n_1).$$

(Note that $\alpha \in [0,\infty]$, $\beta \in [0,2\pi)$.)

Then

$$F_R(r) = \int_0^r \int_0^{2\pi} \frac{\alpha}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(\alpha\cos\beta - A)^2 + (\alpha\sin\beta)^2]} d\beta\, d\alpha$$

$$= \int_0^r \frac{\alpha}{\sigma^2} e^{-\frac{1}{2\sigma^2}(\alpha^2+A^2)} \underbrace{\left[\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\alpha A}{\sigma^2}\cos\beta} d\beta\right]} d\alpha$$

This term is $I_0\left(\frac{\alpha A}{\sigma^2}\right)$ since $I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta$.

Thus

$$F_R(r) = \int_0^r \frac{\alpha}{\sigma^2} e^{-\frac{1}{2\sigma^2}(\alpha^2+A^2)} I_0\left(\frac{\alpha A}{\sigma^2}\right) d\alpha.$$

The probability density function of R

(5.35) (Cont.)

follows by differentiating the cumulative distribution function and is

$$f_R(r) = \frac{dF_R(r)}{dr} = \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+A^2)} I_0\left(\frac{rA}{\sigma^2}\right).$$

(5.36)

$$z(t) = A\cos(\omega_c+\omega_d)t + n(t)$$

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

with PSD of $n(t) = S_n(f) = \begin{cases} N_0/2 & |f-f_c| \le \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$

Note also that the PSD of $n_c(t)$, $n_s(t)$ is $S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0 & |f| \le B/2 \\ 0 & \text{otherwise} \end{cases}$

and $n_c(t)$, $n_s(t)$ are uncorrelated.

(a) $n_c'(t)\cos(\omega_c+\omega_d)t - n_s'(t)\sin(\omega_c+\omega_d)t$

$= n_c'(t)[\cos\omega_c t\cos\omega_d t - \sin\omega_c t\sin\omega_d t]$

$\quad - n_s'(t)[\sin\omega_c t\cos\omega_d t + \cos\omega_c t\sin\omega_d t]$

$= [n_c'(t)\cos\omega_d t - n_s'(t)\sin\omega_d t]\cos\omega_c t$

$\quad - [n_c'(t)\sin\omega_d t + n_s'(t)\cos\omega_d t]\sin\omega_c t$

$= n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t.$

Therefore,

$$n_c'(t)\cos\omega_d t - n_s'(t)\sin\omega_d t = n_c(t)$$

$$n_c'(t)\sin\omega_d t + n_s'(t)\cos\omega_d t = n_s(t).$$

These two equations can now be solved to give expressions for $n_c'(t)$ and $n_s'(t)$.

(5.36) (Cont.)

They are

$$n_c'(t) = n_c(t)\cos\omega_d t + n_s(t)\sin\omega_d t$$

$$n_s'(t) = -n_c(t)\sin\omega_d t + n_s(t)\cos\omega_d t.$$

To find the power spectral densities of $n_c'(t)$, $n_s'(t)$ which are denoted $S_{n_c'}(f)$ and $S_{n_s'}(f)$, we find the autocorrelation functions $R_{n_c'}(\tau)$ and $R_{n_s'}(\tau)$ and Fourier transform.

$$R_{n_c'}(\tau) = E\{n_c'(t)\,n_c'(t+\tau)\}$$

$$= E\{[n_c(t)\cos\omega_d t + n_s(t)\sin\omega_d t][n_c(t+\tau)\cdot\cos\omega_d(t+\tau) + n_s(t+\tau)\sin\omega_d(t+\tau)]$$

$$= R_{n_c}(\tau)\cos\omega_d t\cos\omega_d(t+\tau) + R_{n_s}(\tau)\sin\omega_d t\sin\omega_d(t+\tau)$$

Noting that $R_{n_c}(\tau) = R_{n_s}(\tau)$ and applying appropriate trig. identities we obtain

$$R_{n_c'}(\tau) = R_{n_c}(\tau)\cos\omega_d\tau.$$

In a similar manner it can be shown that

$$R_{n_s'}(\tau) = R_{n_s}(\tau)\cos\omega_d\tau.$$

Next we apply the modulation theorem to obtain the power spectral densities

$$S_{n_c'}(f) = S_{n_s'}(f) = \tfrac{1}{2}S_{n_c}(f+f_d) + \tfrac{1}{2}S_{n_c}(f+f_d).$$

(5) We find the cross power spectral density $S_{n_c'n_s'}(f)$ by first finding the cross-correlation function $R_{n_c'n_s'}(\tau)$ and

(5.36) (Cont.)

then Fourier transforming.

$$R_{n_c' n_s'}(\tau) = E\{n_c'(t)\, n_s'(t+\tau)\}$$

$$= E\{[n_c(t)\cos\omega_d t + n_s(t)\sin\omega_d t]$$

$$[-n_c(t+\tau)\sin\omega_d(t+\tau) + n_s(t+\tau)\cos\omega_d(t+\tau)]\}$$

$$= -R_{n_c}(\tau)\cos\omega_d t \sin\omega_d(t+\tau)$$

$$+ R_{n_s}(\tau)\sin\omega_d t\cos\omega_d(t+\tau).$$

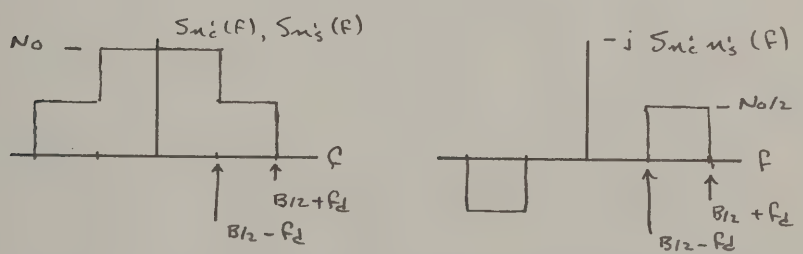
Again note $R_{n_c}(\tau) = R_{n_s}(\tau)$ and so applying trig identities gives

$$R_{n_c' n_s'}(\tau) = -R_{n_s}(\tau)\sin\omega_d\tau.$$

The cross power spectral density is

$$S_{n_c' n_s'}(f) = j\left[\tfrac{1}{2}S_{n_s}(f-f_d) - \tfrac{1}{2}S_{n_s}(f+f_d)\right]$$

The plots of these power spectra are shown below.

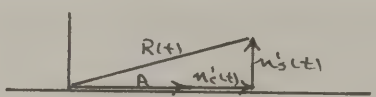


(5.37)

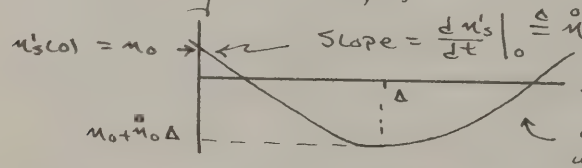
The derivation here follows the following reference:
S.O. Rice "Noise in FM Receivers", pp. 491-504, Proceedings of the Symposium of Time Series Analysis, M. Rosenblatt (ed.) New York: Wiley (1963).
We need additional clicks due to modulation. Thus, consider

$$Z(t) = A\cos 2\pi(f_c+f_d)t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$
$$= A\cos 2\pi(f_c+f_d)t + n_c'(t)\cos 2\pi(f_c+f_d)t - n_s'(t)\sin 2\pi(f_c+f_d)t$$

where from prob. (5-36) $\overline{n_c'(t)\, n_s'(t)} = 0$.
The appropriate phasor diagram is shown below.



The prob. of counter-clockwise origin encirclement in a small time interval Δ is
$P_{ccw\Delta} = P_r\{A + n_c'(t) < 0$ and $n_s'(t)$ undergoes a +to- zero crossing in $(0, \Delta)\}$.
What is the condition for $n_s'(t)$ to undergo a + to - zero crossing in $(0, \Delta)$?



So the condition is that $n_0 > 0$ and $n_0 + \dot{n}_0\Delta < 0$, all $\dot{n}_0 < 0$.
$\therefore P_{ccw\Delta} = P_r\{A + n_c'(0) < 0,\ n_s'(0) > 0,\ n_s'(0) + \dot{n}_s'(0)\Delta < 0$ all $\dot{n}_s' < 0\}$
Similarly, clockwise encirclement prob. is
$P_{cw\Delta} = P_r\{A + n_c'(0) < 0,\ n_s'(0) < 0,\ n_s'(0) + \dot{n}_s'(0)\Delta > 0$ all $\dot{n}_s' > 0\}$
Thus we need the joint pdf of $n_c'(0) \triangleq x$, $n_s'(0) \triangleq y$,

(5.37) (Cont.)

and $\left.\frac{d n_3'}{dt}\right|_{t=0} \triangleq z$. We can show that

$$\overline{x^2} = \overline{y^2} = N_0 B \triangleq a \quad ; \quad \overline{xy} = \overline{yz} = 0$$

$$\overline{z^2} = \tfrac{1}{3}(\pi^2 N_0 B^3) + 4\pi^2 f_d^2 N_0 B \triangleq b + a\omega_d^2 \quad ; \quad \overline{xz} = -a\omega_d .$$

The joint pdf of x, y, z is

$$f_{xyz}(x,y,z) = f_x(x) f_y(y) f_{z/xy}(z/x,y) = f_x(x) f_y(y) f_{z/x}(z/x)$$

$$= \frac{1}{(2\pi)^{3/2} a\sqrt{b}} e^{-\left[\frac{x^2+y^2}{2a} + \frac{(z+x\omega_d)^2}{2b}\right]}$$

$$P_{ccA} = P_r\{x \le -A, \ 0 < y < z\Delta, \ \text{all } z < 0\}$$

$$= \int_{-\infty}^{-A} \int_{-\infty}^{0} \int_{0}^{z\Delta} f_{xyz}(x,y,z)\, dy\, dz\, dx$$

which reduces to

$$P_{ccA} = \Delta\frac{\beta}{2}\left\{\sqrt{1+\alpha^2}\ \operatorname{erfc}\sqrt{\rho(1+\alpha^2)} - \alpha e^{-\rho} \operatorname{erf}\alpha\sqrt{\rho}\right\}$$

where $\alpha = \omega_d/(2\pi\beta)$, $\beta = \sqrt{\frac{b}{a}}\frac{1}{2\pi}$, $2\rho = A^2/a$.

Also, it follows that P_{cA} is given by this expression, but with $\omega_d \to -\omega_d$. Thus

$$\frac{P_{cA}}{\Delta} = \frac{P_{ccA}}{\Delta} + f_d e^{-\rho} \Rightarrow \frac{P_{cA}}{\Delta} > \frac{P_{ccA}}{\Delta} \quad (\text{for } f_d > 0.)$$

This verifies the discussion on pp 249-250.

Putting in values for α, β, ρ gives

$$\frac{P_{ccA}}{\Delta} = \frac{B}{4\sqrt{3}}\left\{\sqrt{1+12\left(\frac{f_d}{B}\right)^2}\ \operatorname{erfc}\sqrt{\left[1+12\left(\frac{f_d}{B}\right)^2\right]\frac{A^2}{2N_0B}}\right.$$

Therefore,

$$V_{mod} = \frac{1}{\Delta}(P_{cA} + P_{ccA}) = \frac{2P_{ccA}}{\Delta} + f_d e^{-A^2/2N_0B}, \ f_d > 0.$$

To make this applicable to square wave modulation, replace f_d by $|f_d|$. Then

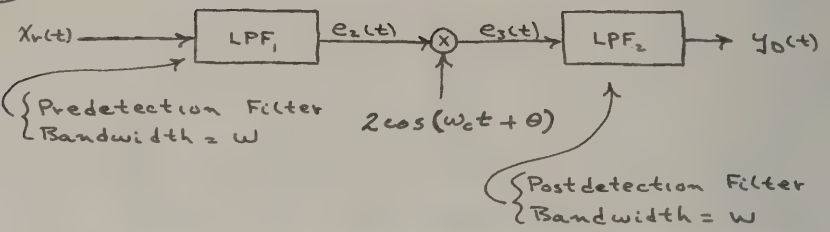
$$V_{mod} = \frac{2P_{ccA}}{\Delta} + |f_d| e^{-A^2/2N_0B}.$$

Note that for $\left|\frac{f_d}{B}\right|$ small this can be written

$$V + \bar{V}$$

where $V = \frac{B}{2\sqrt{3}} \operatorname{erfc}\sqrt{\frac{A^2}{2N_0B}}$, $\bar{V} = |f_d| e^{-A^2/2N_0B}$.

(6.1)



$$x_r(t) = A_c\, m(t)\cos(\omega_c t + \theta) + A_c\, \hat{m}(t)\sin(\omega_c t + \theta) + n(t)$$

$$e_2(t) = A_c\, m(t)\cos(\omega_c t + \theta) + A_c\, \hat{m}(t)\sin(\omega_c t + \theta) + n_c(t)\cos(\omega_x t + \theta) - n_s(t)\sin(\omega_x t + \theta)$$

where $\omega_x = \omega_c - \frac{1}{2}(2\pi W)$

$$\overline{n_c^2} = \overline{n_s^2} = N_0 W.$$

$e_2(t)$ has noise power $N_T = \frac{1}{2}\overline{n_c^2} + \frac{1}{2}\overline{n_s^2} = N_0 W.$

$e_2(t)$ has signal power S_T

$$S_T = A_c^2\,\overline{[A_c m(t)\cos(\omega_c t + \theta) + \hat{m}(t)\sin(\omega_c t + \theta)]^2}$$

$$= \tfrac{1}{2}A_c^2\left[\overline{m^2(t)} + \overline{\hat{m}^2(t)}\right] = A_c^2\,\overline{m^2(t)}.$$

$$\therefore \quad (SNR)_T = \frac{A_c^2\,\overline{m^2(t)}}{N_0 W}$$

$$y_D(t) = L_P[2e_2(t)\cos(\omega_c t + \theta)]$$

$$= A_c\, m(t) + n_c(t)\cos \pi W t - n_s(t)\sin \pi W t,$$

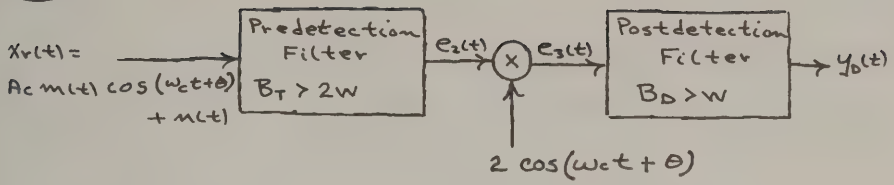
where $L_P[\cdot]$ represents the lowpass portion of the quantity in brackets.

Thus, the signal power is $A_c^2\,\overline{m^2(t)}$ and the noise power is $N_0 W$.

$\therefore$

$$\therefore \quad (SNR)_D = \frac{A_c^2\,\overline{m^2(t)}}{N_0 W} \quad \text{and} \quad \frac{(SNR)_D}{(SNR)_T} = 1.$$

(6.2)



$e_2(t)$ has signal power $\frac{1}{2} A_c^2 \overline{m^2}$.

$e_2(t)$ has noise power $= \overline{n^2} = N_0 B_T$.

Thus, the predetection SNR is $(SNR)_T = \frac{A_c^2 \overline{m^2}}{2 N_0 B_T}$.

The low pass portion of $e_3(t)$ is

$$L_p[e_3(t)] = A_c m(t) + n_c(t).$$

The output signal power is $A_c^2 \overline{m^2}$. The noise power spectral density of $n_c(t)$ is N_0. (See Fig. 5.14).

Case 1: $B_D > \frac{1}{2} B_T$. For this case, the predection filter determines the noise power at $y_D(t)$, N_D.

$N_D = \int_{-\frac{1}{2}B_T}^{\frac{1}{2}B_T} N_0 \, df = N_0 B_T$. Thus detection gain is

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 \overline{m^2}}{N_0 B_T} \Big/ \frac{A_c^2 \overline{m^2}}{2 N_0 B_T} = 2.$$

Case 2: $B_D < \frac{1}{2} B_T$. For this case the post-detection filter limits noise passed by the pre-detection filter and thus determines noise power.

$N_D = \int_{-B_D}^{B_D} N_0 \, df = 2 N_0 B_D$ and so

$$\frac{SNR_D}{SNR_T} = \frac{A_c^2 \overline{m^2}}{2 N_0 B_D} \Big/ \frac{A_c^2 \overline{m^2}}{2 N_0 B_T} = \frac{B_T}{B_D}.$$

(6.3)

This problem is exactly like Problem 6.2 except that the predetection signal power is

$$S_T = \tfrac{1}{2} A_c^2 [1 + a^2 \overline{m_n^2}],$$

and the postdetection signal power is

$$S_D = A_c^2 a^2 \overline{m_n^2}.$$

The noise powers do not change.

Case 1: $B_D > \tfrac{1}{2} B_T$.

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2 \overline{m_n^2}}{N_0 B_T} \Big/ \frac{A_c^2 [1 + a^2 \overline{m_n^2}]}{2 N_0 B_T} = \frac{2 a^2 \overline{m_n^2}}{1 + a^2 \overline{m_n^2}}.$$

Case 2: $B_D < \tfrac{1}{2} B_T$.

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2 \overline{m_n^2}}{2 N_0 B_D} \Big/ \frac{A_c^2 [1 + a^2 \overline{m_n^2}]}{2 N_0 B_T} = \frac{a^2 \overline{m_n^2}}{1 + a^2 \overline{m_n^2}} \frac{B_T}{B_D}$$

(6.4)

Given $a = 0.5$, and $m(t) = 60 \cos 8\pi t$.

$m_n(t) = \cos 8\pi t$, $\overline{m_n^2} = \tfrac{1}{2}$.

From Eqn. 6.19 we have

$$\frac{(SNR)_D}{(SNR)_T} = \frac{2 a^2 \overline{m_n^2}}{1 + a^2 \overline{m_n^2}} = \frac{2 (0.5)^2 (1/2)}{1 + (0.5)^2 (1/2)}$$

$$= 0.222 = -6.53 \text{ dB}.$$

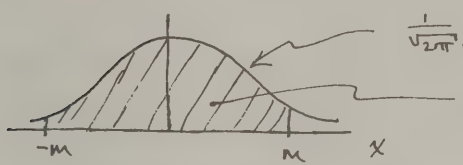
Thus, for the parameters given, AM is inferior to DSB by 9.53 dB.

(6.5)

First we must determine the effective peak value of $m(t)$, M. Since $P[|m(t)| \le M] = 0.99$ we have (see figure)

$$\int_{-M}^{M} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}x^2/\sigma^2} dx = 2\int_{0}^{M} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}x^2/\sigma^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{M/\sqrt{2}\sigma} e^{-y^2} dy = \operatorname{erf}\left(\frac{M}{\sqrt{2}\sigma}\right) = 0.99 .$$



Thus $\frac{M}{\sqrt{2}\sigma} = 1.82$ and $m_n(t) = \frac{m(t)}{1.82\sqrt{2}\sigma}$

$\overline{m_n^2} = \overline{m^2(t)} / (2.574\sigma)^2 = 0.151$ and

$a^2 \overline{m_n^2} = (0.7)^2 (0.151) = 0.0740$.

From Eqn. 6.19 we have

$$\frac{(SNR)_0}{(SNR)_T} = \frac{2a^2 \overline{m_n^2}}{1 + a^2 \overline{m_n^2}} = \frac{2(0.0740)}{1 + 0.0740} = 0.138 .$$

(6.6)

The output of the predetection filter is

$$x_r(t) = A_c[1 + a\, m_n(t)] \cos(\omega_c t + \theta) + n_c(t) \cos(\omega_c t + \theta) - n_s(t) \sin(\omega_c t + \theta)$$

$$= A_c[1 + a\, m_n(t)] \cos(\omega_c t + \theta) + r_n(t) \cos[\omega_c t + \phi_n(t) + \theta]$$

The prob. density function for $r_n(t)$ is [see (4.45)]

$$f_{R_n}(r_n) = \frac{r}{N} e^{-r^2/2N}$$

(6.6) (Cont.)

where $N = \frac{1}{2}\overline{n_c^2} + \frac{1}{2}\overline{n_s^2}$.

From the problem statement we have at threshold, $P[R_m \le A_c] = 0.99$. Thus,

$$0.99 = \int_0^{A_c} \frac{1}{N} x e^{-x^2/2N}\, dx = 1 - e^{-A_c^2/2N}.$$

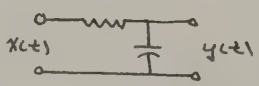
Thus, $e^{-A_c^2/2N} = 0.01 \Rightarrow \frac{A_c^2}{2N} = 4.6$

The predetection signal power is

$$P_T = \frac{1}{2} A_c^2 \left[1 + a^2 \overline{m_m^2}\right] = A_c^2$$

Thus $P_T/2N = 4.6 \Rightarrow P_T/N = 9.2 = 9.63$ dB.

(6.7)



$$H(f) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}.$$

The signal component of $y(t)$ is

$$y_s(t) = \frac{A}{\sqrt{1+\omega^2R^2C^2}} \cos(\omega t - \tan^{-1}\omega RC)$$

and the signal power is

$$S_y = \frac{A^2}{2(1+\omega^2R^2C^2)}.$$

The noise power spectral density at $y(t)$ is

$$N_y(f) = \frac{N_0}{2}|H(f)|^2 = \frac{N_0}{2(1+\omega^2R^2C^2)}$$

and the noise power is

$$N_y = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{1}{1+(2\pi fRC)^2}\, df = \frac{N_0}{2\pi RC}\int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{N_0}{4RC}$$

Thus the signal-to-noise ratio of $y(t)$ is

$$SNR = \frac{S_y}{N_y} = \frac{2A^2RC}{N_0(1+\omega^2R^2C^2)}$$

(6.8)

From (6.19) we have

$$(SNR)_D = \frac{2a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}}(SNR)_T .$$

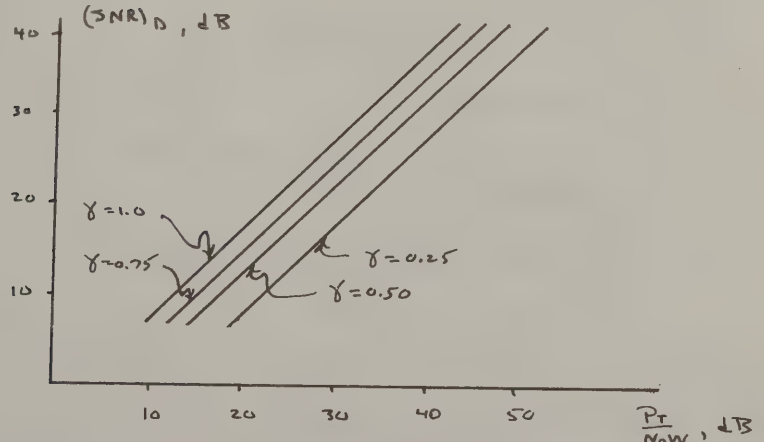
Since $(SNR)_T = P_T/(2N_oW)$ we have

$$(SNR)_D = \frac{a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}}\frac{P_T}{N_oW} .$$

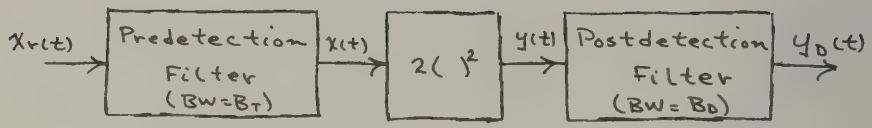
Note that the preceding equation is valid for a coherent detector with any value of P_T/N_oW and is valid for an envelope detector with P_T/N_oW sufficiently large to place operation above threshold. Define the parameter $\gamma = a\sqrt{\overline{m_n^2}}$. Then

$$(SNR)_D = \frac{\gamma^2}{1+\gamma^2}\cdot\frac{P_T}{N_oW} .$$

This definition is made so that a general figure can be constructed. The figure follows.



(6.9)



The predetection bandwidth is defined as B_T while the postdetection bandwidth is defined as B_D.

The received signal is

$$x_r(t) = A_c[1 + a\, m_n(t)] \cos \omega_c t + n(t)$$

where $n(t)$ denotes a white gaussian noise process.

The waveform $x(t)$ is bandpass and so can be written in terms of inphase and quadrature components. Thus

$$x(t) = A_c[1 + a\, m_n(t)] \cos \omega_c t + n_c(t) \cos \omega_c t - n_s(t) \sin \omega_c t.$$

The signal $y(t)$ is

$$\begin{aligned} y(t) = {} & 2A_c^2[1 + a\, m_n(t)]^2 \cos^2 \omega_c t \\ & + 4A_c[1 + a\, m_n(t)]\, n_c(t) \cos^2 \omega_c t \\ & - 4A_c[1 + a\, m_n(t)]\, n_s(t) \cos \omega_c t \sin \omega_c t \\ & + 2n_c^2(t) \cos^2 \omega_c t \\ & - 4n_c(t) n_s(t) \cos \omega_c t \sin \omega_c t \\ & + 2n_s^2(t) \sin^2 \omega_c t. \end{aligned}$$

Assuming that the lowpass postdetection filter bandwidth, B_D, is greater than or equal to

(6.9) (Cont.)

predetection filter bandwidth, B_T, so that terms of the form $m_n^2(t)$, $m_n(t)n_c(t)$, $n_c^2(t)$, and $n_s^2(t)$ are passed, $y_D(t)$ is

$$y_D(t) = A_c^2[1+am_n(t)]^2 + 2A_c[1+am_n(t)]n_c(t) + n_c^2(t) + n_s^2(t)$$

$$= \{A_c[1+am_n(t)] + n_c(t)\}^2 + n_s^2(t)$$

which is $r^2(t)$ in Eqn. 6.28. With $m_n(t) = \cos\omega_m t$ $y_D(t)$ becomes

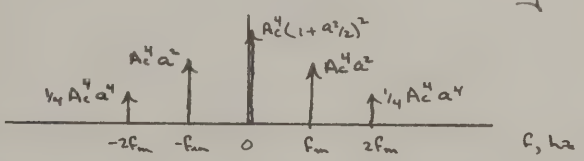
$$y_D(t) = A_c^2[1+a\cos\omega_m t] + 2A_c[1+a\cos\omega_m t]n_c(t) + n_c^2(t) + n_s^2(t)$$

The first term is

$$y_1 = A_c^2[1 + 2a\cos\omega_m t + a^2\cos^2\omega_m t]$$

$$= A_c^2[(1+a^2/2) + 2a\cos\omega_m t + \tfrac{a^2}{2}\cos 2\omega_m t]$$

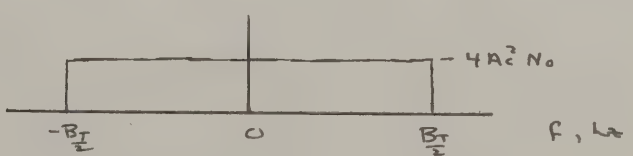
and has the power spectral density (PSD)



The second term is

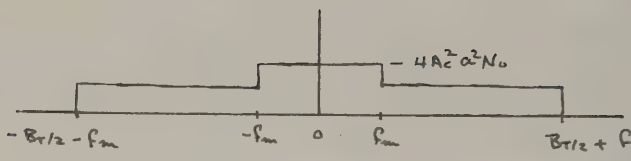
$$y_2 = 2A_c n_c(t) + 2A_c a\, n_c(t)\cos\omega_m t .$$

The first term in this expression has PSD

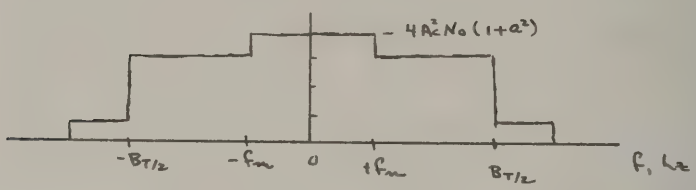


and the second term in this expression has PSD

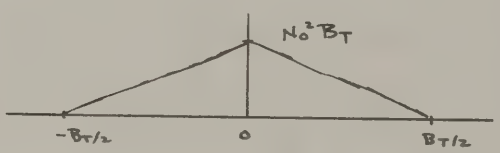
(6.9) (Cont.)



So that the PSD of y_2 is



The third and fourth terms of the expression for $y_D(t)$ has the same PSD which is



(6.10)

For the linear envelope detector we have, from Eqn. 6.19

$$(SNR)_{DL} = \frac{2a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}}(SNR)_T .$$

Since $(SNR)_T = P_T/2N_0W$, $(SNR)_{DL}$ is

$$(SNR)_{DL} = \frac{a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}} P_T/N_0W$$

which for $a=1$, is

$$(SNR)_{DL} = \frac{\overline{m_n^2}}{1+\overline{m_n^2}} P_T/N_0W .$$

We shall now compare this to the square-law detector at high predetection SNR.

(6.10) (Cont)

For sinusoidal modulation the result is (6.37) which, for index $a = 1$ is

$$(SNR)_{DQ} = 2\left(\tfrac{1}{3}\right)^2 \frac{P_T}{N_0 W} = \frac{2}{9}\frac{P_T}{N_0 W}$$

The result for a linear detector, assuming sinusoidal modulation ($\overline{m_n^2} = 1/2$) is

$$(SNR)_{DL} = \frac{1/2}{1+1/2}\frac{P_T}{N_0 W} = \frac{1/2}{3/2}\frac{P_T}{N_0 W} = \frac{1}{3}\frac{P_T}{N_0 W}$$

Thus

$$\frac{(SNR)_{DQ}}{(SNR)_{DL}} = \frac{2/9}{1/3} = 0.667 \rightarrow -1.76\ dB$$

(6.11) For a linear envelope detector operating above threshold with sinusoidal modulation ($\overline{m_n^2} = 1/2$)

$$(SNR)_{DL} = \frac{\frac{1}{2}a^2}{1+\frac{1}{2}a^2}\frac{P_T}{N_0 W} = \frac{a^2}{2+a^2}\frac{P_T}{N_0 W}$$

For large $(SNR)_T$, the result for the square-law detector is, from (6.37)

$$(SNR)_{DQ} = 2\left(\frac{a}{2+a^2}\right)^2 \frac{P_T}{N_0 W}$$

To compare we take the ratio. This gives

$$\frac{(SNR)_{DL}}{(SNR)_{DQ}} = \frac{a^2/(2+a^2)}{2a^2/(2+a^2)^2} = \frac{1}{2}(2+a^2)$$

Thus, the two detectors are essentially equivalent for small index ($a \ll 1$). As the index is increased, the improvement provided by the linear envelope detector becomes more significant. At unity index the linear envelope detector is better by a factor of $\frac{3}{2}$ or 1.76 dB. This is the case of Prob. 6.10.

(6.12)

$$P\{-\infty < M_n \le -1\} + P\{1 < M_n \le \infty\} = 2P\{1 < M_n \le \infty\}$$

$$= 2\frac{1}{\sqrt{2\pi}\sigma}\int_1^{\infty} e^{-v^2/2\sigma^2}\,dv = 0.01$$

therefore

$$\frac{1}{\sqrt{2\pi}\sigma}\int_1^{\infty} e^{-v^2/2\sigma^2}\,dv = 0.005$$

Let $u = v/\sqrt{2}\sigma$ then

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{1/\sqrt{2}\sigma}^{\infty} e^{-u^2}\sqrt{2}\sigma\,du = 0.005$$

$$\frac{1}{\sqrt{\pi}}\int_{1/\sqrt{2}\sigma}^{\infty} e^{-u^2}\,du = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{2}\sigma}\right) = 0.005$$

$$\operatorname{erfc}(1/\sqrt{2}\sigma) = 0.01$$

Thus $\frac{1}{\sqrt{2}\sigma} = 1.822$, $\sigma = \frac{1}{\sqrt{2}(1.822)}$

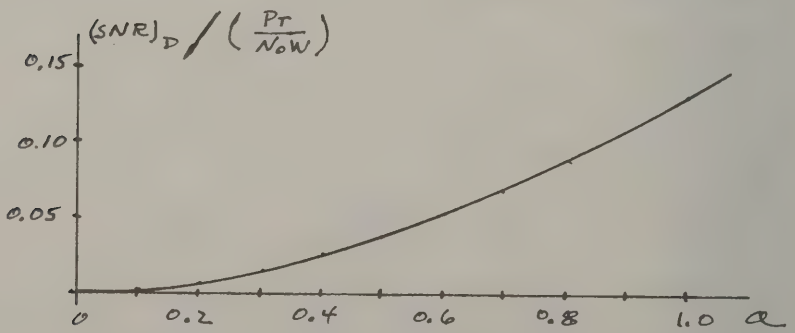
$$\sigma = 0.388$$

$$\sigma^2 = 0.1505$$

Since $m_n(t)$ is zero-mean $\overline{m_n^2} = \sigma^2$.

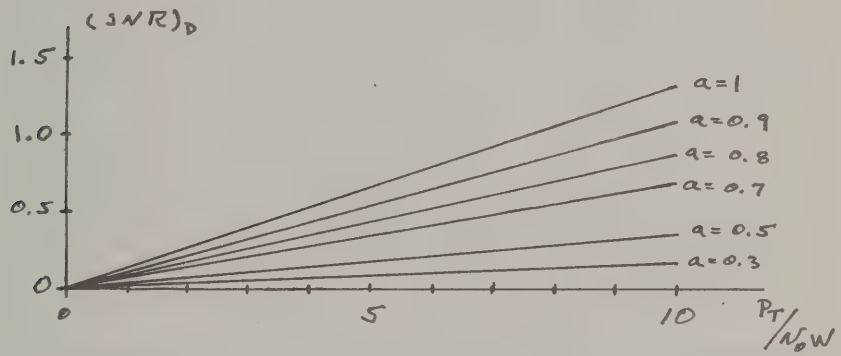
Thus, from (6.21)

$$(SNR)_D = \frac{a^2(0.1505)}{1 + a^2(0.1505)}\left(\frac{P_T}{N_0 W}\right)$$



(6.12) (cont)

The plot could be presented as a plot of $(SNR)_D$ with $\frac{P_T}{N_0 W}$ as the independent variable and the index "a" as a parameter. This would be straight lines with slope $0.1505a^2/(1+0.1505a^2)$, as shown below.



(6.13)

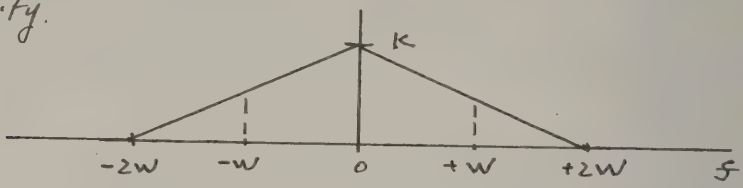
A detailed analysis of the square-law detector is contained in the textbook:

H. Taub and D.L. Schilling, "Principles of Communication Systems", Mc Graw Hill, 1971, pp. 282-288.

The complete analysis will not be presented here. We will compare the noise power output with bandwidth $2W$ to the noise power

(6.13) (cont)

output with postdetection bandwidth W. If $n_c(t)$ and $n_s(t)$ have the power spectral density $\frac{1}{2}N_0\,\Pi\left(\frac{f}{2W}\right)$. $n_c^2(t)$ and $n_s^2(t)$ have the power spectral density.



$$I_1 = 2\int_0^W K\left(1-\tfrac{1}{2W}f\right)df = 2KW - \frac{K}{W}\frac{1}{2}W^2 = \frac{3}{2}KW$$

$$I_2 = 2\int_0^{2W} K\left(1-\tfrac{1}{2W}f\right)df = 4KW - \frac{K}{W}\frac{1}{2}4W^2 = 2KW$$

The ratio of I_1 to I_2 is

$$\frac{I_1}{I_2} = \frac{3/2\,KW}{2\,KW} = \frac{3}{4}$$

Thus the power in $n_c^2(t)$, $\overline{n_c^4}$ is reduced by 25% if the bandwidth is W rather than 2W. Thus $\overline{n_c^4} = \frac{9}{4}N_T$.

$$N_D = 2A_c^2\left(2+a^2\right)N_T + \tfrac{3}{2}N_T^2$$

$$(SNR)_D = \frac{A_c^2a^2/N_T}{(2+a^2) + \frac{3}{2}(N_T/A_c^2)}$$

(6.14)

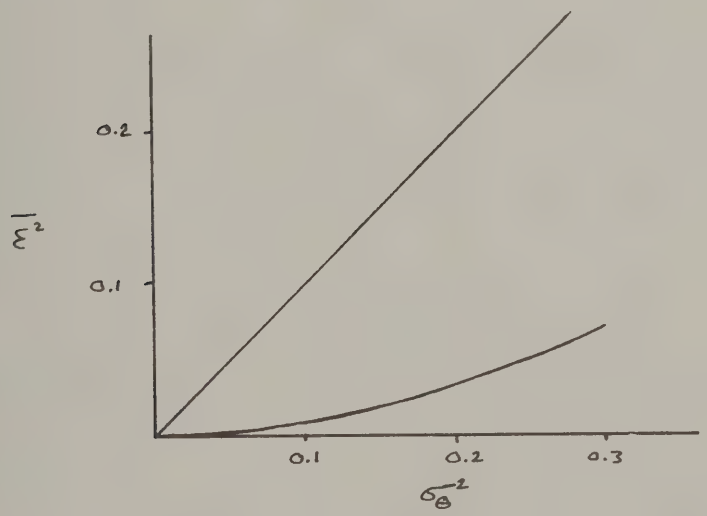
From (6.48) the normalized mean-square error due to noncoherence of the demodulation carrier ($\sigma_N^2 = 0$) for SSB and QDSB is

$$\overline{\varepsilon_{NQ}^2} = \sigma_\theta^2 \qquad \theta \ll 1$$

and the equivalent result for DSB is, from (6.53)

$$\overline{\varepsilon_{ND}^2} = \frac{3}{4}\sigma_\theta^4 \qquad \theta \ll 1.$$

These curves are plotted below for the range of values $0 \le \sigma_\theta^2 \le 0.3$.



(6.14) (Cont.)

These curves show that DSB is much less sensitive to demodulation carrier phase errors than SSB or QDSB. The student should keep in mind that the equations used are valid only for small phase errors.

(6.15)

From (6.48) and (6.53) we have for $\bar{\varepsilon}^2 = 0.01$

$$0.01 = \sigma_\phi^2 + \sigma_N^2/\sigma_m^2 \qquad \text{(for SSB)}$$

and

$$0.01 = \tfrac{3}{4}\sigma_\phi^2 + \sigma_N^2/\sigma_m^2 \qquad \text{(for DSB)}.$$

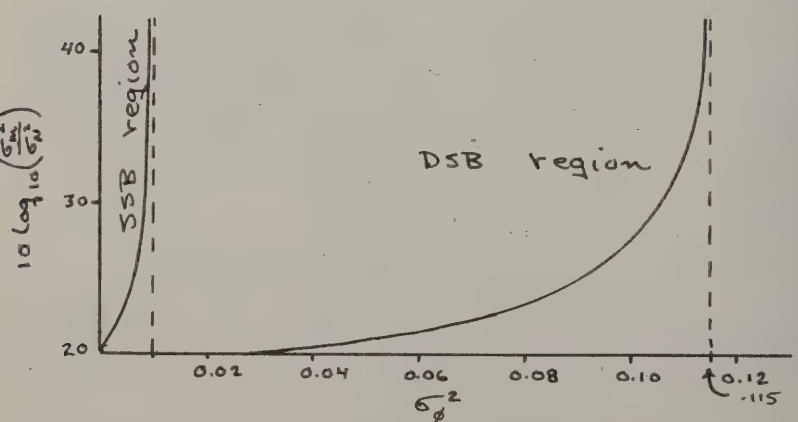
Solving for σ_m^2/σ_N^2 gives

$$\frac{\sigma_m^2}{\sigma_N^2} = \frac{1}{0.01 - \sigma_\phi^2} \qquad \text{(for SSB)}$$

and

$$\frac{\sigma_m^2}{\sigma_N^2} = \frac{1}{0.01 - 0.75\sigma_\phi^4} \qquad \text{(for DSB)}.$$

In the plots below, σ_m^2/σ_N^2 is expressed in dB.



(6.15) (Cont.)

The region of satisfactory performance is the region above and to the left of the respective curve. In this region $\overline{\varepsilon_N^2} < 0.01$. The much larger region for DSB is another way of illustrating that DSB is relatively insensitive to demodulation phase errors if a mean-square error criterion is used.

(6.16)

From (6.53), $\overline{\varepsilon_{ND}^2} = \frac{3}{4}\sigma_\phi^4 + \frac{\sigma_N^2}{\sigma_M^2}$.

Let the bandwidth of the predetection filter be B_T. Then the bandwidth of the pilot filter is $B_P = B_T/\alpha$. Thus, the second term in the equation above is

$$\frac{\sigma_N^2}{\sigma_M^2} = \frac{N_0 B_T}{\sigma_M^2} = \frac{B_T}{B_P} \cdot \frac{N_0 B_P}{\sigma_M^2} = \frac{\alpha}{\rho}$$

since ρ is the pilot SNR and the pilot and message powers are assumed equal. The first term is, using (6.57) and (6.56)

$$\frac{3}{4}\sigma_\theta^4 = \frac{3}{4}\left[\frac{1}{K^2}\frac{1}{2\rho}\right]^2 = \frac{3}{16}\frac{1}{K^4\rho^2}.$$

Thus

$$\overline{\varepsilon_{ND}^2} = \frac{3}{16}\frac{1}{K^4\rho^2} + \frac{\alpha}{\rho}.$$

For the values $K=2$ and $\rho = 20\,dB = 100$,

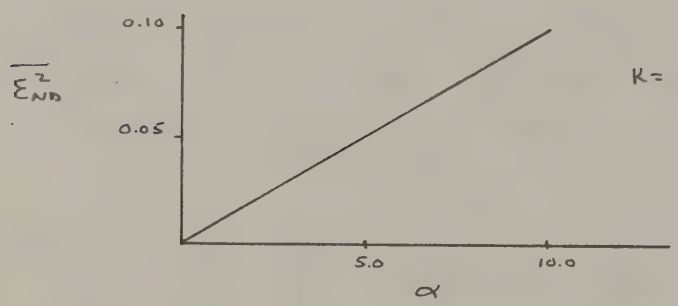
(6.16) (Cont.)

$$\overline{\varepsilon_{ND}^2} = 1.17 \times 10^{-6} + \alpha \times 10^{-2},$$

and for values $K=4$ and $p=100$,

$$\overline{\varepsilon_{ND}^2} = 7.32 \times 10^{-8} + \alpha \times 10^{-2}.$$

In a practical system the predetection filter bandwidth exceeds the pilot filter bandwidth. Thus the error due to demodulation carrier phase errors is negligible compared to the error due to additive noise on a DSB system with a 20dB pilot SNR. Thus $\overline{\varepsilon_{ND}^2} \approx \alpha \times 10^{-2}$.



(6.17)

The normalized mean-square error for the SSB system is, from (6.48),

$$\overline{\varepsilon_{NQ}^2} = \sigma_\phi^2 + \frac{\sigma_N^2}{\sigma_M^2}.$$

As in the previous problem

$$\frac{\sigma_N^2}{\sigma_M^2} = \alpha/p$$

and, using (6.57) and (6.56),

$$\sigma_\phi^2 = \frac{1}{K^2}\frac{1}{2\rho}.$$

(6.17) (Cont.)

Thus

$$\overline{\varepsilon_{NQ}^2} = \frac{1}{2PK^2} + \frac{\alpha}{P} = \frac{1}{P}\left(\alpha + \frac{1}{2K^2}\right).$$

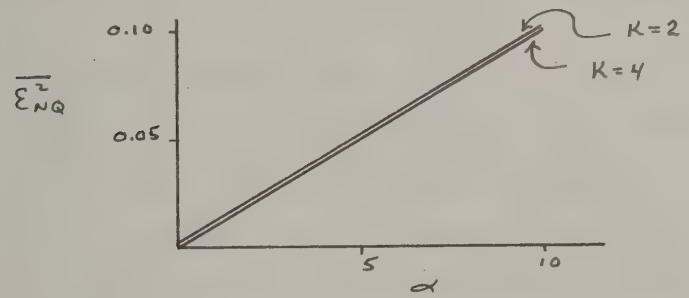
For the values $K = 2$ and $p = 20\,dB = 100$

$$\overline{\varepsilon_{NQ}^2} = (0.125 + \alpha) \times 10^{-2}$$

and for $K = 4$ and $p = 100$

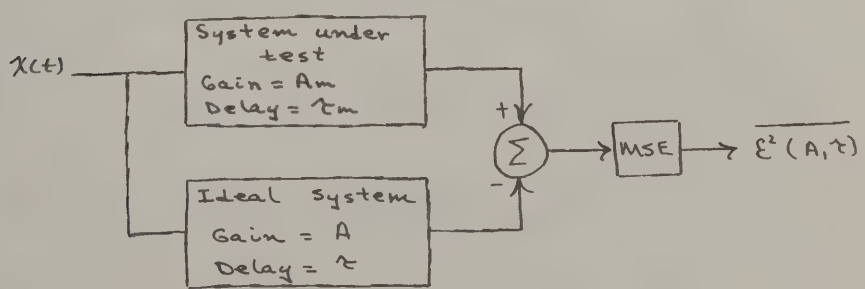
$$\overline{\varepsilon_{NQ}^2} = (0.031 + \alpha) \times 10^{-2}.$$

Note that, unlike the DSB result, we can no longer assume that the errors due to demodulation carrier phase errors are negligible. The results are plotted below.



(6.18)

The conceptual test is shown below.



(6.18) (Cont.)

The parameters A and τ are adjusted until $\overline{\varepsilon^2(A,\tau)}$ is a minimum. The mean-square error is

$$\overline{\varepsilon^2(A,\tau)} = \overline{[y(t) - A\,x(t-\tau)]^2}$$

$$= \overline{y^2(t)} - 2A\,\overline{x(t-\tau)\,y(t)} + A^2\,\overline{x^2(t-\tau)}.$$

By definition of autocorrelation

$$\overline{y^2(t)} = R_y(0) \quad \text{and} \quad \overline{x^2(t-\tau)} = R_x(0).$$

By definition of cross-correlation

$$\overline{x(t-\tau)\,y(t)} = \overline{x(t)\,y(t+\tau)} = R_{xy}(\tau).$$

Thus

$$\overline{\varepsilon^2(A,\tau)} = R_y(0) + A^2 R_x(0) - 2A R_{xy}(\tau).$$

It can be seen from the expression for $\overline{\varepsilon^2(A,\tau)}$ that for any fixed gain A, $\overline{\varepsilon^2(A,\tau)}$ is minimized by maximizing $R_{xy}(\tau)$. Thus, under the definitions of A_m and τ_m, $\tau = \tau_m$ is the value of τ which maximizes the system input-output cross correlation and is the system time delay. With τ_m fixed we have

$$\overline{\varepsilon^2(A,\tau_m)} = R_y(0) + A^2 R_x(0) - 2A R_{xy}(\tau_m)$$

and

$$\frac{\partial\,\overline{\varepsilon^2(A,\tau_m)}}{\partial A} = 2A R_x(0) - 2A R_{xy}(\tau_m) = 0$$

which gives

$$A_m = \frac{R_{xy}(\tau_m)}{R_x(0)}$$

and

(6.18) (Cont.)

$$\overline{\varepsilon^2(A_m,\tau_m)} = R_y(0) + \left[\frac{R_{xy}(\tau_m)}{R_x(0)}\right]^2 R_x(0) - 2\frac{R_{xy}(\tau_m)}{R_x(0)} R_{xy}(\tau_m)$$

$$= R_y(0) - \frac{R_{xy}^2(\tau_m)}{R_x(0)}$$

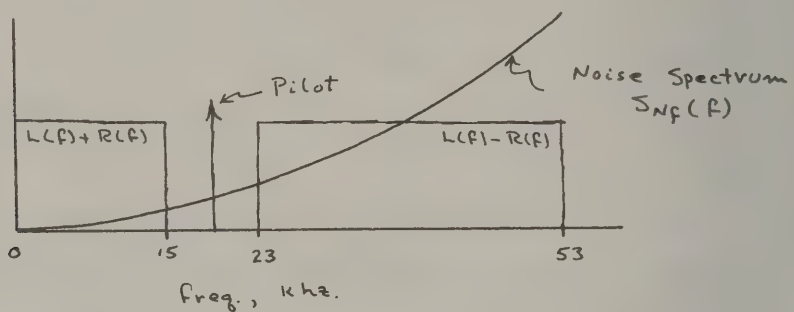
which is the required result. This is clearly a minimum since $\overline{\varepsilon^2(A,\tau)}$ can be made arbitrarily large by increasing A.

Note:

The gain and delay of a linear system is often defined as the magnitude of the system transfer function and the slope of the phase characteristic, respectively. The definition of gain and delay suggested by this problem is useful for defining system gain when the magnitude of the transfer function is not constant and for defining delay when the phase characteristic is not linear over a frequency range of interest. It should also be noted that the definition of gain and delay used in this problem can be used even when the system is nonlinear, a situation for which the transfer function (magnitude and phase) is not even defined.

(6.19)

From Prob. 3.54, we know that the spectrum of the FM discriminator output, with stereophonic broadcasting, is as shown below.



From (6.83) we know that the noise spectrum is given by

$$S_{Nf}(f) = \frac{K_D^2}{A_c^2} N_0 f^2.$$

The predetection noise powers are easily computed. For the $l(t)+r(t)$ channel

$$P_{N,L+R} = 2\int_0^{15000} \frac{K_D^2}{A_c^2} N_0 f^2 df = 2.25\times10^{12}\,\frac{K_D^2}{A_c^2} N_0$$

and for the $(l(t)-r(t)$ channel

$$P_{N,L-R} = 2\int_{23000}^{53000} \frac{K_D^2}{A_c^2} N_0 f^2 df = 91.14\times10^{12}\,\frac{K_D^2}{A_c^2} N_0.$$

Thus, the predetection noise power in the $l(t)-r(t)$ channel is over 40 times the noise power in the $l(t)+r(t)$ channel. After demodulation the factor will be approximately 20 because of the 3dB detection gain in the coherent demodulation of the $l(t)-r(t)$ channel.

(6.19) (Cont.)

We have shown that the main source of noise is in the $l(t) - r(t)$ channel. Thus, in high noise environments, monophonic broadcasting is to be preferred over stereophonic broadcasting.

(6.20)

The discriminator output signal power is, from (6.77)

$$S_{DF} = K_D^2 f_d^2 \overline{m^2}$$

The output noise power with preemphasis is, from (6.94)

$$N_{DF} = 2 \frac{K_D^2}{A_c^2} N_0 f_3^3 \left(\frac{W}{f_3} - \tan^{-1} \frac{W}{f_3} \right).$$

Thus

$$(SNR)_{DF} = \frac{S_{DF}}{N_{DF}} = \frac{K_D^2 f_d^2 \overline{m_n^2}}{2 \frac{K_D^2}{A_c^2} N_0 W f_3^2 \left(1 - \frac{f_3}{W} \tan^{-1} \frac{W}{f_3} \right)}$$

$$= \frac{1}{1 - \frac{f_3}{W} \tan^{-1} \frac{W}{f_3}} \left(\frac{f_d}{f_3} \right)^2 \overline{m_n^2} \frac{P_T}{N_0 W}.$$

With the values used in Example 6.1 ($f_d = 75$ KHz, $W = 15$ KHz, $f_3 = 2.1$ KHz, and $\overline{m_n^2} = 0.1$) we have

$$(SNR)_{DF} = \frac{1}{1 - \frac{2.1}{15} \tan^{-1} \frac{15}{2.1}} \left(\frac{75}{2.1} \right)^2 (0.1) \frac{P_T}{N_0 W}$$

(6.20) (Cont.)

which is

$$(SNR)_{DF} = 160 \frac{P_T}{N_0 W}.$$

This is 22 dB superior to baseband. Thus the error in making the approximation

$$\left(1 - \frac{f_3}{W} \tan^{-1} \frac{W}{f_3}\right) \cong 1$$

is 20% or 1 dB with the given values of f_3 and W.

(6.21)

The discriminator output SNR is given by

$$(SNR)_{DF} = \left(\frac{f_d}{f_3}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W}.$$

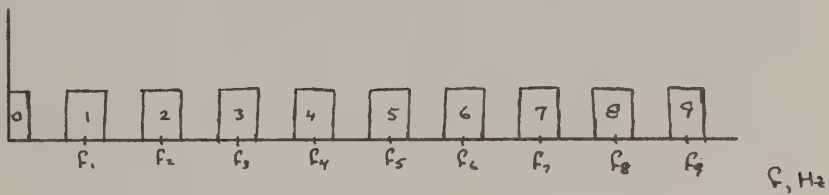
With $f_d = 75$ KHz., $f_3 = 1.5$ KHz, and $\overline{m_n^2} = 0.1$ this becomes

$$(SNR)_{DF} = \left(\frac{75}{1.5}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W} = 250 \frac{P_T}{N_0 W}.$$

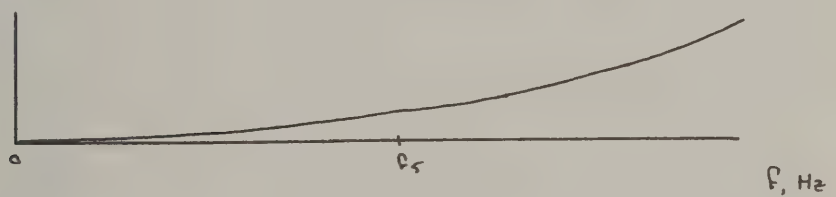
It can be seen that reducing f_3 results in a significantly increased postdetection SNR assuming operation above threshold. However, decreasing f_3 results in increased amplification of the high frequency components of $m(t)$. If $m(t)$ has significant high frequency content, the result is increased transmitter deviation requiring increased transmission bandwidth. This increased bandwidth may place operation too close to threshold unless P_T is increased.

(6.22)

The received FDM signal spectrum is



and the noise spectrum is



The K^{th} channel signal is given by

$$x_K(t) = A_K m_K(t) \cos 2\pi K f_1 t \qquad K = 0, 1, 2, \ldots, 9.$$

The spectrum of this signal occupies the frequency range (note that $f_K = 4KW$

$$(4K-1)W \le f \le (4K+1)W.$$

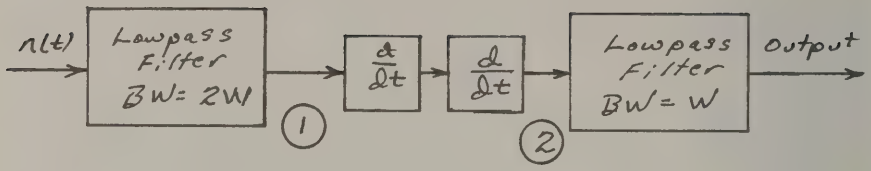
Thus the predetection noise power for the K^{th} channel is proportional to

$$N_K = \int_{(4K-1)W}^{(4K+1)W} f^2 df = \frac{W^3}{3}\left[(4K+1)^3 - (4K-1)^3\right]$$

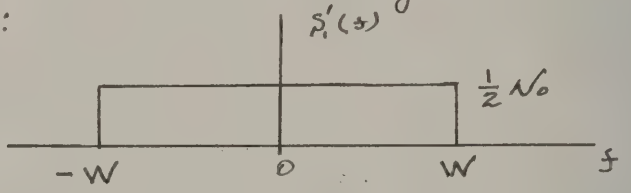
$$= \tfrac{1}{3} W^3 [96K^2 + 2] \approx 32 W^3 K^2 \quad \text{for } K \gg 1$$

The signal power is proportional to A_K^2. Thus the SNR is given by $(SNR)_D = \lambda A_K^2 / K^2$ where λ is a constant of proportionality and is a function of W, K_0, A_c and N_0. If all channels are to have the same $(SNR)_D$, A_K must be a linear function of K.

(6.23)



The Power Spectral Density at point (1) is:



The transfer function from (1) to (2) is

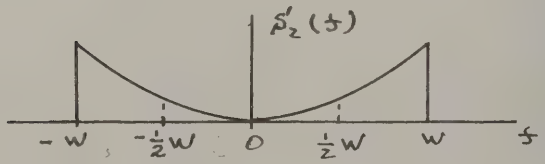
$$H(f) = (j2\pi f)^2 = -4\pi^2 f^2$$

so that

$$|H(f)| = 4\pi^2 f^2$$

The power spectral density at (2) is

$$S_2(f) = \begin{cases} 2\pi^2 f^2 N_0, & |f| < W \\ 0, & |f| > W \end{cases}$$



The power spectral density at the output is

$$S_0(f) = \begin{cases} 2\pi^2 f^2 N_0, & |f| < \frac{1}{2}W \\ 0, & |f| > \frac{1}{2}W \end{cases}$$

The output power P_0 is

$$P_0 = \int_{-W/2}^{W/2} 2\pi^2 f^2 N_0 \, df = 2\pi^2 N_0 (2) \frac{1}{3} \left(\frac{W}{2}\right)^3 = \frac{\pi^2 N_0 W^3}{6}$$

(6.24)

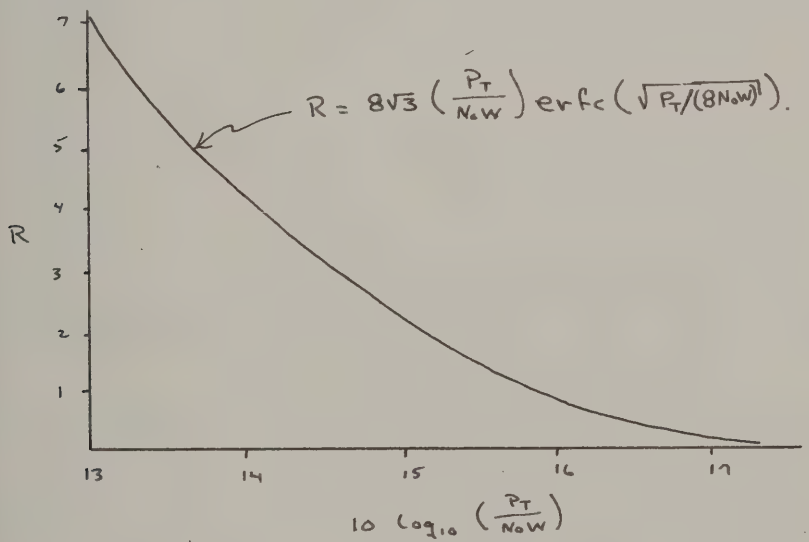
Let the ratio of spike noise power to Gaussian noise power be denoted R. Thus, from Eqn. 6.104

$$R = \frac{\text{Spike Noise Power}}{\text{Gaussian Noise Power}} = \frac{\frac{1}{\sqrt{3}} K_D^2 B_T W \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2 N_o B_T}}\right)}{\frac{2}{3} \frac{K_D^2}{A_c^2} N_o W^3}.$$

Since $B_T \approx 2(D+1)W$ and $P_T = \frac{1}{2}A_c^2$ R becomes

$$R = 2\sqrt{3}\,(D+1)(P_T/N_oW) \operatorname{erfc}\left(\sqrt{P_T/[2(D+1)N_oW]}\right)$$

This is plotted below for P_T/N_o expressed in dB, and is for D=3.



The spike noise power is equal to the Gaussian noise power at approximately $P_T/N_oW = 16$ dB. For a fixed value of P_T/N_oW, increasing D increases R. Thus, increasing D moves the R=1 point to higher values of P_T/N_oW. In

(6.24) (Cont.)

other words, the threshold point moves to the right with increasing values of the deviation ratio, D.

It should be noted that as P_T/N_oW is reduced, the spike noise power quickly becomes more important. For example, at $P_T/N_oW = 18$ dB spike noise is negligible, and at $P_T/N_oW = 14$ dB spike noise predominates.

(6.25)

Equation 6.111 is

$$S_\phi(f) = (2\pi)^2 \left[\frac{B_T}{2\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2N_oB_T}}\right) + \frac{2f_d}{\pi} \exp\left(-\frac{A_c^2}{2N_oB_T}\right)\right]$$

Since $B_T \approx 2(\beta+1)W$ and $f_d = \beta W$, this can be written

$$S_\phi(f) = (2\pi)^2 \left[\frac{(\beta+1)W}{\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{1}{2(\beta+1)}\frac{P_T}{N_oW}}\right) + \frac{2\beta W}{\pi} \exp\left(\frac{-1}{2(\beta+1)}\frac{P_T}{N_oW}\right)\right]$$

which, for $\beta = 3$, can be written

$$S_\phi(f) = (2\pi)^2 W \left[\frac{4}{\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{1}{8}\frac{P_T}{N_oW}}\right) + \frac{6}{\pi} \exp\left(-\frac{1}{8}\frac{P_T}{N_oW}\right)\right].$$

Since only the relative amplitudes of the two terms are of interest

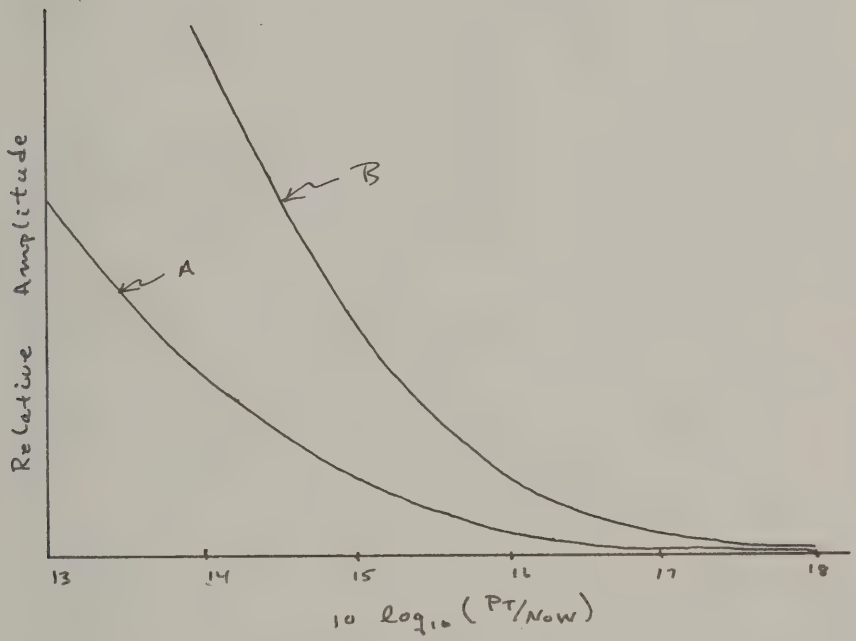
$A = \frac{4}{\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{1}{8}\frac{P_T}{N_oW}}\right)$ and $B = \frac{6}{\pi} \exp\left(-\frac{1}{8}\frac{P_T}{N_oW}\right)$ will be plotted.

The results of the last problem show that, considering only the first term, A, threshold is in the region of $\frac{P_T}{N_oW} = 16$ dB. The second term, B, adds to the noise power and thus

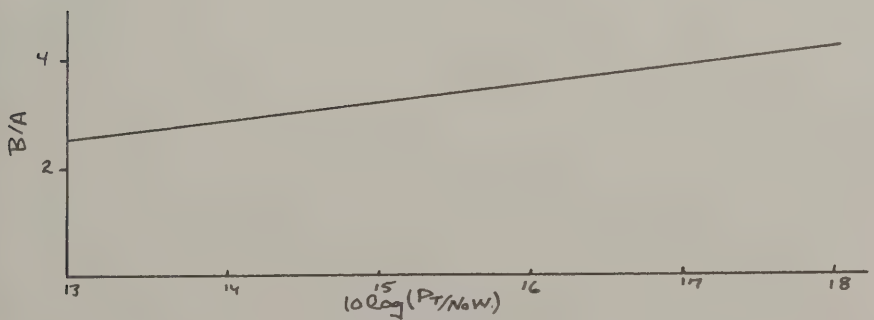
(6.25) (Cont.)

will raise the value of $\frac{P_T}{NoW}$ at which threshold takes place.

The two terms, A and B, are plotted below. As expected, B > A.



Since this curve is difficult to read for large $\frac{P_T}{NoW}$, the ratio $\frac{B}{A}$ is plotted and shown below.



(6.25) (Cont.)

It can be seen that $A+B \approx 5A$ over the range 16-18 dB. Multiplying R by 5 in the previous problem shows that the result will be to move the R=1 point from 16 dB to 17 dB. Thus, with $\beta=3$, the effect of modulation is to increase the threshold value of $\frac{P_T}{N_0 W}$ by approximately 1 dB.

Another significant observation is that the second term in Eqn. 6.107 is only moderately larger than the first term. It can be seen from the plot of B/A that the difference between the two terms decreases as $\frac{P_T}{N_0 W}$. At sufficiently small values of P_T/N_0W, $A>B$. However, at these values of P_T/N_0W, the system is operating well below threshold and the results are of little interest.

To fully understand the role of D, R in Problem 6.24 should be plotted for several values of D, and the relative amplitudes of A and B in Problem 6.25 should be determined for different values of β.

(6.26)

The instantaneous frequency deviation in Hz. is given by

$$\delta f = f_d\, m(t) \triangleq x(t)$$

where $x(t)$ is a zero-mean Gaussian random process with variance $\sigma_x^2 = f_d^2 \sigma_m^2$. Thus

$$\overline{|\delta f|} = \overline{|x|} = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}x^2/\sigma_x^2}\, dx$$
$$= 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x}\, x\, e^{-\frac{1}{2}x^2/\sigma_x^2}\, dx .$$

Using change of variables with $y = \frac{x}{\sqrt{2}\sigma_x}$ gives

$$\overline{|\delta f|} = \overline{|x|} = 2\int_0^{\infty} \frac{1}{\sqrt{\pi}}\, y\, e^{-y^2} \sqrt{2}\sigma_x\, dy = \sqrt{\frac{2}{\pi}}\sigma_x$$
$$= \sqrt{\frac{2}{\pi}} f_d \sigma_m$$

where $\int_0^{\infty} s\, e^{-s^2} ds = \frac{1}{2}$ has been used. Substitution into Eqn. 5.88 yields

$$\overline{\delta v} = \sqrt{\frac{2}{\pi}} f_d \sigma_m \exp\left(\frac{-A_c^2}{2N_0 B_T}\right)$$

where B_T is the bandwidth of the predetection filter. Thus

$$S_\phi(f) = (2\pi)^2 \left[\frac{B_T}{2\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2N_0 B_T}}\right) + \sqrt{\frac{2}{\pi}} f_d \sigma_m \exp\left(\frac{-A_c^2}{2N_0 B_T}\right)\right].$$

As in the text, we shall assume that the second term predominates over the first term so that the spike noise power at the output of the postdetection filter is

$$N_{DS} = K_D^2 (2W) \sqrt{\frac{2}{\pi}} f_d \sigma_m \exp\left(\frac{-A_c^2}{2N_0 B_T}\right).$$

Combining this with the output noise power due to Gaussian noise yields

(6.26) (cont.)

$$N_D = \frac{2}{3}\left(\frac{K_D}{A_c}\right)^2 N_0 W^3 + 2W K_D^2 \sqrt{\frac{2}{\pi}} f_d \sigma_m \exp\left(-\frac{A_c^2}{N_0 B_T}\right).$$

Since $S_D = K_D^2 f_d^2 \sigma_m^2$ we have

$$(SNR)_D = \frac{f_d^2 \sigma_m^2}{\frac{1}{3}\left[\frac{N_0 W}{P_T}\right] W^2 + 2W\sqrt{\frac{2}{\pi}} f_d \sigma_m \exp\left[\frac{-1}{2(D+1)} P_T/N_0 W\right]}$$

where we have used $P_T = \frac{1}{2}A_c^2$ and $B_T \approx 2(D+1)W$. Since $f_d = DW$ (we are assuming that the maximum value of $|m(t)|$ is unity. If this is not the case we shall include the normalizing constant in f_d.) we have

$$(SNR)_D = \frac{3D^2\sigma_m^2\left[P_T/N_0 W\right]}{1 + 6\sqrt{\frac{2}{\pi}} D \sigma_m \left[P_T/N_0 W\right] \exp\left[\frac{-1}{2(D+1)} P_T/N_0 W\right]}.$$

Note: The student should plot $(SNR)_D$ as a function of $\frac{P_T}{N_0 W}$ for several values of D and compare with the sinusoidal case. In order to make such a comparison, let $\sigma_m^2 = 1/2$. The student should also check the approximation that the second term in $S_Y(f)$ predominates over the first term as was done for sinusoidal modulation in Problem 6.25.

(6.27)

From (6.104)

$$N_D = \frac{2}{3}\frac{K_D^2}{A_c^2} N_o W^3 + \frac{K_D^2 B_T W}{\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2N_o B_T}}\right)$$

Equating the two terms on the left-hand side of the above equation yields

$$\frac{2}{3}\frac{N_o W^2}{A_c^2} = \frac{B_T}{\sqrt{3}} \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2N_o B_T}}\right)$$

This gives

$$\frac{B_T}{\sqrt{3}}\frac{3}{2}\frac{A_c^2}{N_o W^2} \operatorname{erfc}\left(\sqrt{\frac{A_c^2}{2N_o B_T}}\right) = 1$$

Since $P_T = \frac{1}{2}A_c^2$, and $K = \frac{B_T}{W}$, we have the desired result

$$K\sqrt{3}\,\frac{P_T}{N_o W} \operatorname{erfc}\left(\sqrt{\frac{P_T}{N_o W}\frac{1}{K}}\right) = 1$$

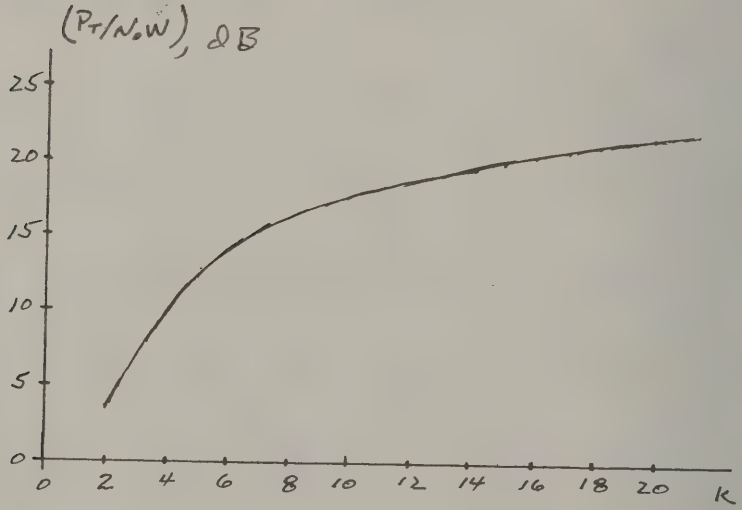
The following table can be constructed

K	$\frac{P_T}{N_o W}$	$\frac{P_T}{N_o W}$, dB	$\frac{P_T}{N_o B_T}$, dB
2	2.45	3.4	0.9
3	7.45	8.7	4.0
5	18.75	12.7	5.7
7	31.75	15.0	6.6
10	53.5	17.3	7.3
15	93.6	19.7	8.0
20	137.5	21.4	8.4

The last column is added to the

(6.27) (Cont)

table in order to compare the threshold values with Figure 6.17. Good agreement with Figure 6.17 can be seen.



(6.28)

We start with (6.123)

$$(SNR)_D = \frac{4\tau_1^2 \overline{m_n^2} T_s B_T W}{\tau_0} \left(\frac{P_T}{N_0 W}\right).$$

First we must solve for τ_0. Since we assume $\overline{m_n} = 0$, we have $\overline{\tau(t)} = \tau_0$. Let τ_0 equal the pulse risetime plus the pulse fall-time, which is simply twice the pulse risetime. Then

$$\tau_0 = 2T_R = 2\left(\frac{1}{2B_T}\right) = \frac{1}{B_T}.$$

If τ_0 is fixed at a given value, then $B_T = \frac{1}{\tau_0}$ is the minimum bandwidth which can be used for pulse transmission. For Nyquist rate sampling

$$T_s = \frac{1}{f_s} = \frac{1}{2W}$$

so that $T_s W = 1/2$. The peak-to-peak pulse displacement is $2\tau_1 = T_s$ which gives, assuming Nyquist rate sampling once again

$$\tau_1 = \frac{1}{2} T_s = \frac{1}{4W}.$$

Letting $\tau_0 = \frac{1}{B_T}$, $T_s W = 1/2$, and $\tau_1 = \frac{1}{2W}$ in the expression for $(SNR)_D$ yields

$$(SNR)_D = \frac{4\left(\frac{1}{4W}\right)^2 \overline{m_n^2} \frac{1}{2} B_T}{1/B_T} \left(\frac{P_T}{N_0 W}\right)$$

or

$$(SNR)_D = \frac{1}{8} \overline{m_n^2} \left(\frac{B_T}{W}\right)^2 \left(\frac{P_T}{N_0 W}\right)$$

(6.28) (Cont.)

which is

$$(SNR)_D = K\left(\frac{B_T}{W}\right)^2 \frac{P_T}{N_0 W}, \quad K = \frac{1}{8}\overline{m_n^2}.$$

Note:

Many assumptions have been made, most of which would not be valid in application. For example, the sampling rate is typically several times the Nyquist rate and typically $B_T \gg \frac{1}{\tau_0}$. However, these assumptions effect only the numerical value of K. The expression

$$(SNR)_D = K\left(\frac{B_T}{W}\right)^2 \frac{P_T}{N_0 W}$$

remains valid assuming operation above threshold (See Chapter 8, Fig. 8.15).

Carlson (1968) extends this problem somewhat by also considering PWM and compares the results with FM.

(6.29)

The signal power is

$$S_Q = \frac{1}{2}(5)^2 = 12.5 \text{ Watts}$$

and, from (6.130), the quantization noise power is

$$N_Q = \frac{1}{12}5^2$$

(6.29) (Cont.)

where S, the width of a quantization level is

$$S = \frac{10}{q} = \frac{10}{2^N}$$

since the range of the A/D converter is 10 units and there are $q = 2^N$ quantization levels, where n is the word-length. Thus,

$$(SNR)_Q = \frac{12.5}{\frac{1}{12}\left(\frac{10}{2^N}\right)^2} = 1.5\,(2^{2N}).$$

The following assumptions have been made:

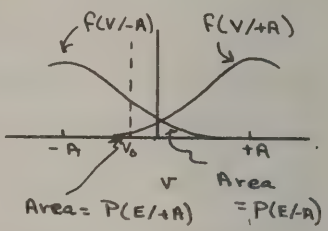
1) Any dc component of the signal has been neglected (assumed zero).

2) The power of the quantized signal is assumed equal to the power of the unquantized signal. This is equivalent to assuming that there are a large number of quantization levels, i.e., n is large.

3) We assumed in deriving (6.126) that the signal is uniformly distributed within a quantization level. Once again, we are assuming that n is large resulting in a large number of quantization levels.

7.1

Designate the output of integrate-and-dump detector as V. Then $V = \begin{cases} AT+N & \text{if } +A \text{ sent} \\ -AT+N & \text{if } -A \text{ sent} \end{cases}$ where N is Gaussian, zero mean, $\text{var}\{N\} = \frac{N_0}{2}T$.



(a) $P(\text{Error}/+A \text{ sent}) = P(V < V_0 / +A)$

$P(\text{Error}/-A \text{ sent}) = P(V > V_0 / -A)$

$P(\text{Error}) = P(\text{Error}/+A)\,P(+A) + P(\text{Error}/-A)\,P(-A)$

$$= \tfrac{3}{4} P(V < V_0/+A) + \tfrac{1}{4} P(V > V_0/-A)$$

$$= \frac{3}{4}\int_{-\infty}^{V_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}(V-AT)^2/\sigma^2\right] dV + \frac{1}{4}\int_{V_0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}(V+AT)^2/\sigma^2\right] dV$$

where $\sigma^2 = \frac{N_0}{2}T$.

(b) Minimize P(Error) with respect to V_0 by taking derivative and setting to zero.

$$\frac{d\,P(\text{Error})}{d\,V_0} = \frac{3}{4}\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}(V_0-AT)^2/\sigma^2\right] - \frac{1}{4}\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}(V_0+AT)^2/\sigma^2\right]$$

$$= 0$$

Thus $\frac{3}{4} \exp\left[-\tfrac{1}{2}(V_0-AT)^2/\sigma^2\right] = \frac{1}{4} \exp\left[-\tfrac{1}{2}(V_0+AT)^2/\sigma^2\right]$

Taking natural log gives

$$2\cdot\ln 3 - (V_0-AT)^2/\sigma^2 = -(V_0+AT)^2/\sigma^2$$

Solving for V_0 gives $V_0 = -\frac{N_0 \ln 3}{4A}$.

Note that V_0 is always negative so we choose +A more often. This is reasonable since +A occurs more often. Also, when N_0 gets large we always choose +A, as we should to minimize P(Error).

7.2

$A = 1.0$, $N_0 = 10^{-4}$, V_0 = threshold = 0,
$P(\text{Error}) \leq 10^{-6}$

(a) From (7.11) $P(\text{Error}) = \frac{1}{2}\operatorname{erfc}\left(A\sqrt{T/N_0}\right)$ with $\operatorname{erfc}(x) \approx \frac{1}{x\sqrt{\pi}} e^{-x^2}$ solve for x so that $\operatorname{erfc}(x) \leq 2\times10^{-6} \Rightarrow 2.0\times10^{-6} = \frac{1}{x\sqrt{\pi}} e^{-x^2}$
$3.55\times10^{-6}\ e^{-x^2}$ solving for x iteratively gives

$$x = 2.37$$

$$A\sqrt{T/N_0} \geq 2.37 \Rightarrow 10^2\sqrt{T} \geq 2.37 \Rightarrow T \geq 1.14\text{ ms}$$

(b) $A\sqrt{T/N_0}$ is constant so $A \to \sqrt{10} = 3.16$

(c) A must increase by $\sqrt{2}$, $A \to \sqrt{2}$

7.3

(a) The impulse response of the filter is $2\pi f_3 e^{-2\pi f_3 t} u(t)$. We desire the filter output at time T for a unit step input of amplitude A. Thus, we convolve $Au(t)$ with $h(t)$ and evaluate at T:

$$S_0(T) = \int_0^T 2\pi f_3 e^{-2\pi f_3 \alpha} A u(T-\alpha)\, d\alpha = A\left[1 - e^{-2\pi f_3 T}\right]$$

The power spectral density at the filter output is

$$S_n'(f) = \frac{N_0}{2}\left|\frac{1}{1+jf/f_3}\right|^2 = \frac{N_0}{2}\frac{1}{1+(f/f_3)^2}$$

$$E\{N^2\} = R_n(\tau)\big|_{\tau=0} = \mathcal{F}^{-1}\{S_n'(f)\}$$

$$= N_0\int_0^\infty \frac{1}{1+(f/f_3)^2}\, df = \frac{N_0\pi f_3}{2} = N_0 W$$

where W is the equivalent noise bandwidth of the filter. Thus

$$\frac{S_0^2(T)}{E\{N^2\}} = \frac{2A^2\left[1-e^{-2\pi f_3 T}\right]^2}{N_0\pi f_3}$$

7.3 (cont.)

(b) To maximize $\frac{S_o^2(T)}{E\{N^2\}}$ we differentiate with respect to f_3 and set it equal to zero. Then solve for f_3. This yields

$$\frac{\partial}{\partial f_3}\frac{S_o^2(T)}{E\{N^2\}} = \frac{N_o \pi f_3 4A^2\left[1-e^{-2\pi f_3 T}\right](2\pi T e^{-2\pi f_3 T}) - 2A^2\left[1-e^{-2\pi f_3 T}\right]^2 N_o \pi}{(N_o \pi f_3)^2}$$

Setting this equal to zero and solving for f_3 yields

$$4\pi f_3 T\, e^{-2\pi f_3 T} - 1 + e^{-2\pi f_3 T} = 0$$

or with $\alpha = 2\pi f_3 T$ we have

$$(2\alpha + 1)e^{-\alpha} = 1$$

Solving $(2\alpha+1)e^{-\alpha}$ for several values of α yields the table

α	$(2\alpha+1)e^{-\alpha}$
1.00	1.10
1.10	1.07
1.20	1.02
1.25	1.00
1.30	0.98

Thus $\alpha = 1.25 = 2\pi f_3 T$ and therefore the value

$$f_3 T = \frac{1.25}{2\pi}$$

is the value which maximizes

$$S_o^2(T)/E\{N^2\}$$

7.4

(a) Denote output due to signal as $s_o(t)$ and output due to noise as $n_o(t)$. Then, we desire to maximize: $[s_o(t_o)]^2 / E\{n_o^2(t)\}$.

$s_o(t_o) = \mathcal{F}^{-1}\{S(f) H_m(f)\}|_{t=t_o} = \int_{-\infty}^{\infty} S(f) H_m(f) e^{j2\pi f t_o} df.$

$E\{n_o^2(t)\} = R_{n_o}(\tau)|_{\tau=0}. \quad R_{n_o}(\tau) = \mathcal{F}^{-1}\left\{\frac{N_o}{2}|H_m(f)|^2\right\}.$

$\therefore E\{n_o^2(t)\} = \frac{N_o}{2}\int_{-\infty}^{\infty}|H_m(f)|^2 df.$

Thus, $$\frac{[s_o(t_o)]^2}{E\{n_o^2(t)\}} = \frac{2}{N_o}\cdot\frac{\int_{-\infty}^{\infty} S(f)H_m(f) e^{j2\pi f t_o} df}{\int_{-\infty}^{\infty} H_m(f) H_m^*(f) df}.$$

Schwarz's inequality states: $\left[\int f(x)g(x)dx\right]^2 \le \int f^2(x)dx \int g^2(x)dx$ with equality when $f(x) = c\,g(x)$.

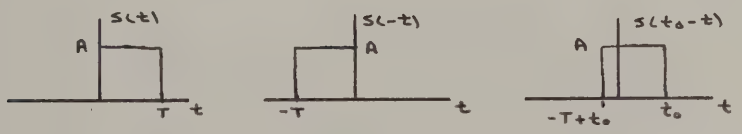
$\therefore$ $$\frac{[s_o(t_o)]^2}{E\{n_o^2(t)\}} \le \frac{2}{N_o}\cdot\frac{\int_{-\infty}^{\infty} S(f) S^*(f) df \cdot \int_{-\infty}^{\infty} H_m(f) H_m^*(f) df}{\int_{-\infty}^{\infty} H_m(f) H_m^*(f) df}$$

$$= \frac{2}{N_o}\int_{-\infty}^{\infty} S(f) S^*(f) df$$

which is a maximum for $H_m(f) = S^*(f) e^{-j2\pi f t_o}$.

(b) $h_m(t) = s(t_o - t)$ by taking Fourier transform of $H_m(f)$ by applying time reversal and time delay theorems.

(c)



For $t_o = 0$: $h_{mr}(t) = 0$ everywhere.

For $t_o = T/2$: $h_{mr}(t) = A$ for $0 \le t \le T/2$, 0 elsewhere

For $t_o = T$: $h_{mr}(t) = A$ for $0 \le t \le T$, 0 elsewhere

For $t_o = 2T$: $h_{mr}(t) = A$ for $T \le t \le 2T$, 0 elsewhere.

7.4 (Cont.)

(d) $S_o(t_o) = s(t) * h_{mr}(t)\big|_{t=t_o} = \int_{-\infty}^{\infty} s(t_o-\alpha)\, h_{mr}(\alpha)\, d\alpha.$

For $t_o = 0$: $S_o(t_o) = 0$

For $t_o = T/2$: $S_o(t_o) = \frac{1}{2}A^2T$

For $t_o = T$: $S_o(t_o) = A^2T$

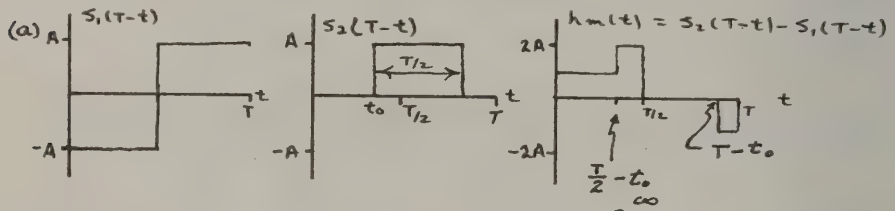
For $t_o = 2T$: $S_o(t_o) = A^2T$

Note that t_o should be larger than signal duration.

7.5

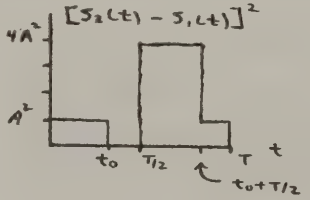
(Assume a realizable matched filter.)

(a)



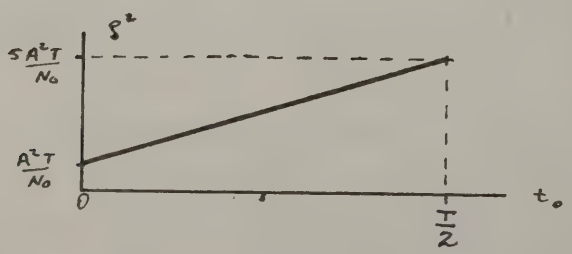
(b) Using Equation 7-37, $\rho^2 = \frac{2}{N_o}\int_{-\infty}^{\infty} [s_2(t) - s_1(t)]^2\, dt$

Graphically:



Therefore

$$\rho^2 = \frac{2}{N_o}\left[\frac{A^2T}{2} + 4A^2t_o\right], \quad 0 \le t_o \le T/2.$$

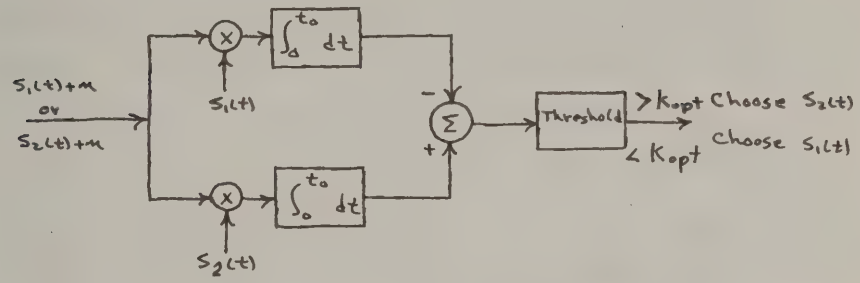


Note that maximum of ρ^2 occurs at $t = T/2$.

(7.5) (Cont.)

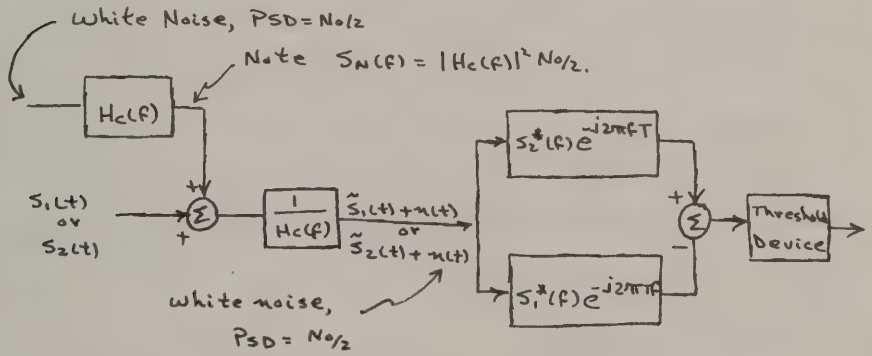
(c) $P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{\delta}{2\sqrt{2}}\right]$, to minimize use maximum δ, so let $t_0 = T/2$, and then $\delta = \sqrt{\frac{SA^2T}{N_0}}$.

(d) Correlator receiver:



(7.6)

Consider the system as shown below:



We can analyze as before for white noise.

$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{\delta}{2\sqrt{2}}\right]$ where $\delta = \left[\frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df\right]^{1/2}$

and $g(t) = \tilde{s}_2(t) - \tilde{s}_1(t)$.

$G(f) = \tilde{S}_2(f) - \tilde{S}_1(f) = \frac{1}{H_c(f)}[S_2(f) - S_1(f)]$, $H_c(f) = \sqrt{\frac{2}{N_0}S_N(f)}$.

$\therefore G(f) = \sqrt{\frac{N_0}{2S_N(f)}}[S_2(f) - S_1(f)]$ and so

$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{1}{2\sqrt{2}}\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{S_N(f)}}|S_2(f) - S_1(f)|^2 df\right\}^{1/2}\right]$.

(7.7)

Let $z = \frac{E}{N_o}$ = SNR. Then the expressions for Pe for each type of transmission are

ASK: $Pe = \frac{1}{2}\,\mathrm{erfc}(\sqrt{z/2})$.

PSK: $Pe = \frac{1}{2}\,\mathrm{erfc}(\sqrt{(1-m^2)z})$ for m^2 = fraction of power in carrier.

FSK: $Pe = \frac{1}{2}\,\mathrm{erfc}(\sqrt{z/2})$.

First lets examine $\mathrm{erfc}(u)$. $\mathrm{erfc}(u) \approx \frac{1}{\sqrt{\pi}u}\, e^{-u^2}$.

Solution by iteration gives

$\mathrm{erfc}(2.20) = 2\times10^{-3}$

$\mathrm{erfc}(2.64) = 2\times10^{-4}$

$\mathrm{erfc}(3.02) = 2\times10^{-5}$

(a) For ASK, FSK $\sqrt{z/2} = 2.20 \Rightarrow z = 9.68 = 9.86$ dB

For PSK $\sqrt{.9z} = 2.20 \Rightarrow z = 5.38 = 7.31$ dB

(b) For ASK, FSK $\sqrt{z/2} = 2.64 \Rightarrow z = 13.94 = 11.44$ dB

For PSK $\sqrt{.9z} = 2.64 \Rightarrow z = 7.74 = 8.89$ dB

(c) For ASK, FSK $\sqrt{z/2} = 3.02 \Rightarrow z = 18.24 = 12.61$ dB

For PSK $\sqrt{.9z} = 3.02 \Rightarrow z = 10.13 = 10.06$ dB.

(7.8)

$z = 11$ db $\Rightarrow z = 12.59$

(a) $Pe = \frac{1}{2}\,\mathrm{erfc}(\sqrt{z/2}) = \frac{1}{2}\,\mathrm{erfc}(2.51) \approx \frac{1}{2(2.51)}\, e^{-2.51^2} = 2.06\times10^{-4}$

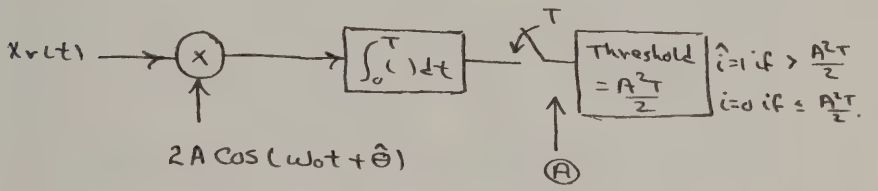
(b) $\sqrt{(1-m^2)(12.59)} = 2.51 \Rightarrow (1-m^2) = 0.5 \Rightarrow m = 0.707$.

(7.9)

The received waveform is

$$x_r(t) = \begin{cases} A\cos(\omega_0 t + \theta) + n(t) & 0 \le t \le T;\ i = 1 \\ n(t) & 0 \le t \le T;\ i = 0 \end{cases}$$

where $n(t)$ is white-Gaussian noise with power spectral density $N_0/2$ and the receiver structure is



First we find the conditional output at point A. Let $i=1$. Denote the output X_1.

$$X_1 = \int_0^T 2A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\hat{\theta})\,dt + \int_0^T n(t)\,2A\cos(\omega_0 t+\hat{\theta})\,dt$$

$$= A^2T\cos(\theta-\hat{\theta}) + N_1$$

where N is Gaussian.

$$\sigma_N^2 = \frac{N_0}{2}\int_0^T 4A^2\cos^2(\omega_0 t+\hat{\theta})\,dt = 4A^2\frac{T}{2}(N_0/2)$$

$$= A^2TN_0.$$

Next let $i=0$. Denote the output X_0.

$$X_0 = \int_0^T n(t)\,2A\cos(\omega_0 t+\hat{\theta})\,dt = N$$

where N is as before.

We find $\Pr\{\text{Error}\}$ using conditional probabilities of error. However, this case is different than most because

(7.9) (Cont.)

the transmitted energy is different for each hypothesis. Thus all conditional error probabilities must be computed. Note that

$$\Pr\{E\} = \Pr\{E/i=1\}\Pr\{i=1\} + \Pr\{E/i=0\}\Pr\{i=0\}.$$

First we compute $\Pr\{E/i=1\}$.

$$\Pr\{E/i=1\} = \Pr\{X_1 < A^2T/2\}$$
$$= \Pr\{A^2T\cos(\theta-\hat{\theta}) + N \le A^2T/2\}$$
$$= \Pr\{N \le A^2T[1/2 - \cos(\theta-\hat{\theta})]\}.$$
$$= \frac{1}{2}\operatorname{erfc}\left\{\frac{A^2T}{\sqrt{2A^2TN_0}}[\cos(\theta-\hat{\theta}) - 1/2]\right\}$$
$$= \frac{1}{2}\operatorname{erfc}\left\{2\sqrt{\frac{z}{2}}[\cos(\theta-\hat{\theta}) - 1/2]\right\}$$

Note that two assumptions have been made in arriving at this last expression. First, we have assumed $\theta - \hat{\theta} \le \cos^{-1}(1/2)$ solely for brevity. Also, we've taken $z = A^2T/4N_0$. This is the case for ASK since signal is only $A\cos\omega_0 t$ for 1/2 the time.

Next we compute $\Pr\{E/i=0\}$.

$$\Pr\{E/i=0\} = \Pr\{N > A^2T/2\} = 1/2\operatorname{erfc}(\sqrt{z/2}).$$

Therefore

$$\Pr\{E\} = 1/4\operatorname{erfc}\left\{2\sqrt{\frac{z}{2}}[\cos(\theta-\hat{\theta}) - 1/2]\right\} + \frac{1}{4}\operatorname{erfc}\{\sqrt{z/2}\}.$$

where $\Pr\{i=0\} = \Pr\{i=1\}$ has been used.

(7.10)

For each case a 1 is used for reference bit.

(a) Message:		1	1	0	1	1	0	0	0	1	1	0
Encoded:	1	1	1	0	0	0	1	0	1	1	1	0
(b) Message:		1	0	1	0	1	0	1	0	1	0	1
Encoded:	1	1	0	0	1	1	0	0	1	1	0	0
(c) Message:		1	1	1	1	1	0	0	0	0	0	0
Encoded:	1	1	1	1	1	1	0	1	0	1	0	1

(7.11)

$$Pe(PRK) = \tfrac{1}{2}\operatorname{erfc}\left(\sqrt{z}\,|\cos\phi|\right) \approx \frac{1}{2\sqrt{\pi z}\,|\cos\phi|}\, e^{-z\cos^2\phi}.$$

$$Pe(DPSK) = \tfrac{1}{2} e^{-z}.$$

$$Pe(DPSK) = 10^{-5} \Rightarrow z = 10.82.$$

$$\therefore Pe(PRK)\big|_{z=10.82} = 10^{-5} \Rightarrow e^{-10.82\cos^2\phi} = 1.17\times10^{-4}\,|\cos\phi|.$$

Solving iteratively gives $\phi = 23^\circ$.

7.12

TYPE OF SIGNALLING	SIGNAL SET	P_e for $z \gg 1$	z in dB for $P_e = 10^{-4}$	
Baseband	$\pm A$	$\frac{1}{2\sqrt{\pi z}} e^{-z}$	8.4	
Coherent ASK	0 $A\cos\omega_c t$	$\frac{1}{\sqrt{2\pi z}} e^{-z/2}$	11.4	
PRK	$\pm A\cos\omega_c t$	$\frac{1}{2\sqrt{\pi z}} e^{-z}$	8.4	
PSK (m^2 in Carrier)	see text	$\frac{e^{-(1-m^2)z}}{2\sqrt{\pi z(1-m^2)}}$	8.9	$m^2 = 0.1$
PRK (with imperfect ref.)	$\pm A\cos\omega_c t$	$\frac{e^{-z\cos^2\phi}}{2\sqrt{\pi z}\lvert\cos\phi\rvert}$	9.1	$\phi = 10^\circ$
FSK	$A\cos\omega_1 t$ $A\cos\omega_2 t$	Same as ASK	11.4	
DPSK	–	$P_e = \frac{1}{2} e^{-z}$	9.3	
Noncoherent ASK	–	$P_e = \frac{1}{2} e^{-z/2}$	12.3	
Noncoherent FSK	–	$P_e = \frac{1}{2} e^{-z/2}$	12.3	

(7.13)

Denote the signalling scheme by

$s_1(t) = A\cos\omega_0 t \quad 0 \le t \le T$

$s_2(t) = A\cos(\omega_0+\Delta\omega)t \quad 0 \le t \le T.$

Then,

$$\sqrt{E_1E_2}\,\rho_{12} = \int_0^T A^2\cos\omega_0 t\cos(\omega_0+\Delta\omega)t\,dt$$

$$= \tfrac{1}{2}A^2\int_0^T[\cos\Delta\omega t + \cos(2\omega_0+\Delta\omega)t]\,dt$$

$$= \tfrac{1}{2}A^2\left[\frac{1}{\Delta\omega}\sin\Delta\omega T + \frac{1}{2\omega_0+\Delta\omega}\sin(2\omega_0+\Delta\omega)T\right]$$

Now assume that ω_0 and T are fixed and we desire to minimize $\sqrt{E_1E_2}\,\rho_{12}$ with respect to $\Delta\omega$. Assume that $2\omega_0$ is large compared to any $\Delta\omega$ we might choose so that the 2ND term is negligible. Then

$$\sqrt{E_1E_2}\,\rho_{12} = \tfrac{1}{2}A^2\left[\frac{1}{\Delta\omega}\sin\Delta\omega T\right] = \tfrac{1}{2}A^2T\,\mathrm{sinc}\left(\frac{\Delta\omega T}{\pi}\right).$$

From tables, the sinc function takes on a minimum of -0.216 for an argument of 1.4.

$\therefore$ minimum of $\sqrt{E_1E_2}\,\rho_{12} = \frac{1}{2}A^2T(-.216)$

and occurs when $\Delta\omega = 1.4\pi/T$.

The Pe with orthogonal signal set is given by

$P_e = \tfrac{1}{2}\,\mathrm{erfc}\left\{\tfrac{1}{2}\left[\frac{E_1+E_2}{N_0}\right]^{1/2}\right\}$ where $E_1=E_2=\tfrac{1}{2}A^2T$.

The Pe with minimum ρ_{12} is given by

$$P_e = \tfrac{1}{2}\,\mathrm{erfc}\left\{\tfrac{1}{2}\left[\frac{E_1+E_2+2(.216)E_1}{N_0}\right]^{1/2}\right\}$$

where $E_1 = E_2 = \tfrac{1}{2}A^2T$.

Thus, the increase in SNR is given by

$\frac{2+.432}{2} = 1.216 = 0.85$ dB.

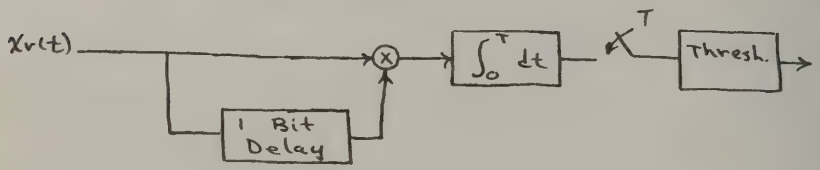
(7.14)

(See Downing, 1964)

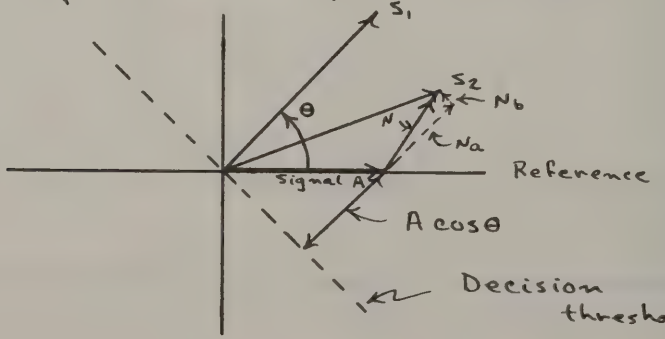
The received waveform for DPSK is

$$X_r(t) = A_c m_i(t) \cos(\omega_c t) + n(t)$$

where the noise waveform $n(t)$ is white Gaussian noise with power spectral density $N_0/2$, and $m_i(t)$ is a ± 1 bit stream and is computed by differentially encoding a data stream. The receiver structure is



The probability of error, $P(E)$, is computed by examining two adjacent inputs. Denote these s_1 and s_2. Then the resulting phasor diagram is shown below.



(7.14) (Cont.)

The total noise N is broken into two components N_a and N_b which are independent and the direction of N_A is useful in $P(E)$ computation. Note that the threshold is set solely by the previous reception S_1. (This "moving threshold" is a result of the DPSK modulation scheme, and does not refer to another fixed reference in the receiver.) Also, A, θ and N are related to E and N_0 as given presently. Also, note from the phasor diagram that an error is made when N_a exceeds $A\cos\theta$.

It can be shown with little difficulty that $A = \sqrt{E}$; N_a is gaussian with $\mu = 0$, $\sigma^2 = N_0$ and θ is a random variable corresponding to the phase of a sinusoid plus additive white gaussian noise with a resulting pdf

$$f_\theta(\theta) = \frac{1}{2\pi} e^{-E/2N_0} + \sqrt{\frac{E}{2\pi N_0}} \cos\theta \, e^{-(E/2N_0)\sin^2\theta} \cdot \int_{-\infty}^{\sqrt{\frac{E}{N_0}}\cos\theta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Now $P(E)$ can be written

$$P(E) = \int_{-\pi}^{\pi} P(E/\theta) f_\theta(\theta) d\theta$$

and

(7.14) (Cont.)

$$P(E/\theta) = Pr\{N_a > A\cos\theta\} = \int_{\sqrt{E/N_0}\cos\theta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx .$$

First we simplify $P(E/\theta)$.

$$P(E/\theta) = \int_{\sqrt{E/N_0}\cos\theta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{E/N_0}\cos\theta} e^{-x^2/2}\,dx .$$

Writing $e^{-x^2/2}$ as a series gives

$$P(E/\theta) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{E/N_0}\cos\theta} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}\,dx$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)} \left(\sqrt{\frac{E}{N_0}}\cos\theta\right)^{2n+1} .$$

Note that $f_\theta(\theta)$ is an even function and $P(E/\theta)$ has an even and odd part given by

$$P(E/\theta)_{even} = 1/2$$

$$P(E/\theta)_{odd} = -\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)} \left(\sqrt{\frac{E}{N_0}}\cos\theta\right)^{2n+1}$$

Now we integrate to find $P(E)$.

$$P(E) = \int_{-\pi}^{\pi} f_\theta(\theta)\left[P(E/\theta)_{even} + P(E/\theta)_{odd}\right] d\theta$$

$$= \int_{-\pi}^{\pi} f_\theta(\theta)\, P(E/\theta)_{even}\, d\theta .$$

since limits are symmetric about the origin. Thus

$$P(E) = \int_{-\pi}^{\pi} \left[\frac{1}{2\pi} e^{-\frac{E}{2N_0}} + \sqrt{\frac{E}{2\pi N_0}}\cos\theta\, e^{-\frac{E}{2N_0}\sin^2\theta} \cdot \int_{-\infty}^{\sqrt{\frac{E}{N_0}}\cos\theta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx\right]\left(\frac{1}{2}\right) d\theta$$

$$= \frac{1}{2} e^{-\frac{E}{2N_0}} + \int_{-\pi}^{\pi} \left[\sqrt{\frac{E}{2\pi N_0}}\cos\theta\, e^{-\frac{E}{2N_0}\sin^2\theta} \cdot \int_{-\infty}^{\sqrt{\frac{E}{N_0}}\cos\theta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx\right]\left(\frac{1}{2}\right) d\theta$$

(7.14) (Cont.)

We show that the second term in this expression is zero. We write

$$\int_{-\infty}^{\sqrt{E/N_0}\cos\theta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \frac{1}{2} + \int_{0}^{\sqrt{E/N_0}\cos\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)} \left(\sqrt{\frac{E}{N_0}}\cos\theta\right)^{2n+1}.$$

The last term is odd and so since $\cos\theta\, e^{-(E/N_0)\sin^2\theta}$ is even this term gives no contribution to integral. Thus

$$P(E) = \frac{1}{2} e^{-E/2N_0} + \int_{-\pi}^{\pi} \frac{1}{4} \sqrt{\frac{E}{2\pi N_0}} \cos\theta\, e^{-\frac{E}{2N_0}\sin^2\theta}\,d\theta.$$

Finally, it can be shown that the last term in this expression is zero. To see this note that $e^{-\frac{E}{2N_0}\sin^2\theta}$ is periodic in π and even about $\pm\pi/2$, while $\cos\theta$ is periodic in 2π and and odd about $\pm\pi/2$. Therefore

$$P(E) = \frac{1}{2} e^{-E/2N_0}.$$

Errors tend to cluster in twos for DPSK because two adjacent symbols are highly correlated.

(7.15)

$N_1 = \int_0^T n(t) \sin \omega_0 t \, dt$, $N_2 = \int_0^T n(t) \cos \omega_0 t \, dt$ where $n(t)$ is white gaussian noise with PSD = $N_0/2$.

$$E\{N_1 N_2\} = E\left\{ \int_0^T n(\alpha) \sin \omega_0 \alpha \, d\alpha \int_0^T n(\beta) \cos \omega_0 \beta \, d\beta \right\}$$

$$= E\left\{ \int_0^T \int_0^T n(\alpha) n(\beta) \sin \omega_0 \alpha \cos \omega_0 \beta \, d\alpha \, d\beta \right\}$$

$$= \int_0^T \int_0^T E\{n(\alpha) n(\beta)\} \sin \omega_0 \alpha \cos \omega_0 \beta \, d\alpha \, d\beta$$

$$= \int_0^T \int_0^T \delta(\alpha - \beta) \sin \omega_0 \alpha \cos \omega_0 \beta \, d\alpha \, d\beta$$

$$= \int_0^T \sin \omega_0 \beta \cos \omega_0 \beta \, d\beta = \int_0^T \tfrac{1}{2} \sin 2\omega_0 \beta \, d\beta$$

$$= 0 \quad \text{because we assume} \quad \omega_0 = 2\pi n T.$$

(7.16)

$$\theta_i = \tan^{-1}\left[\frac{-m_{i2}(t)}{m_{i1}(t)}\right] = \pm 45^\circ, \pm 135^\circ.$$

Case	θ_i	$m_{i1}(t)$	$m_{i2}(t)$	$\hat{\theta}$ for $m_{i2}(t)$ incorrect	$\hat{\theta}$ for $m_{i1}(t)$ incorrect
1	45°	+1	−1	-45°	135°
2	135°	−1	−1	-135°	45°
3	-45°	+1	+1	45°	-135°
4	-135°	−1	+1	135°	-45°
		(a)		(b)	(c)

(7.17)

Note that $R_{m_i}(\tau) = \Lambda(\tau/T)$ is the autocorrelation function of $m_i(t)$, where the bit period is T. Now $m_{i1}(t)$, $m_{i2}(t)$ are derived by serial-to-parallel conversion of $m_i(t)$ and so have bit period 2T. We first need to find the autocorrelation functions of $m_{i1}(t)$, $m_{i2}(t)$ which are denoted $R_{m_{i1}}(\tau)$, $R_{m_{i2}}(\tau)$. We note that these are

$$R_{m_{i1}}(\tau) = R_{m_{i2}}(\tau) = \Lambda(\tau/2T).$$

This follows since $R_{m_i}(\tau)$ results from the random coin toss ± 1 sequence. Note also that

$$R_{m_{i1}, m_{i2}}(\tau) = 0$$

and so $m_{i1}(t)$, $m_{i2}(t)$ are uncorrelated.

(a) The transmitted waveform is

$$X_t(t) = A m_{i1}(t) \sin \omega_c t + A m_{i2}(t) \cos \omega_c t.$$

Thus the autocorrelation function $R_{X_t}(\tau)$ is

$$R_{X_t}(\tau) = E\{X_t(t) X_t(t+\tau)\}$$

$$= E\{[A m_{i1}(t) \sin \omega_c t + A m_{i2}(t) \cos \omega_c t] \cdot [A m_{i1}(t+\tau) \sin \omega_c (t+\tau) + A m_{i2}(t+\tau) \cos \omega_c (t+\tau)]\}$$

$$= A^2 R_{m_{i1}}(\tau) \left[\tfrac{1}{2} \cos \omega_c \tau - \tfrac{1}{2} \cos \omega_c (2t+\tau) + \tfrac{1}{2} \cos \omega_c \tau + \tfrac{1}{2} \cos \omega_c (2t+\tau)\right]$$

$$= A^2 R_{m_{i1}}(\tau) \cos \omega_c \tau.$$

(7.17) (Cont.)

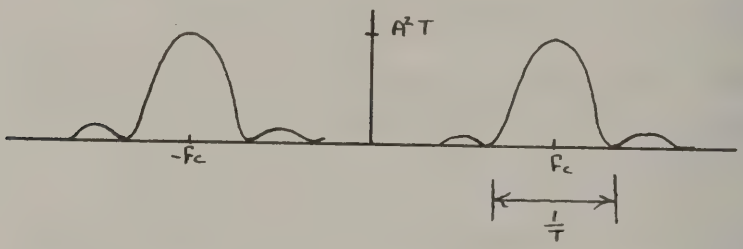
Noting that

$$\mathcal{F}\{R_{mi}(\tau)\} = 2T \operatorname{sinc}^2 2fT$$

and applying the modulation theorem gives

$$S_{x+}(f) = A^2T\left[\operatorname{sinc}^2 2T(f-f_c) + \operatorname{sinc}^2 2T(f+f_c)\right].$$

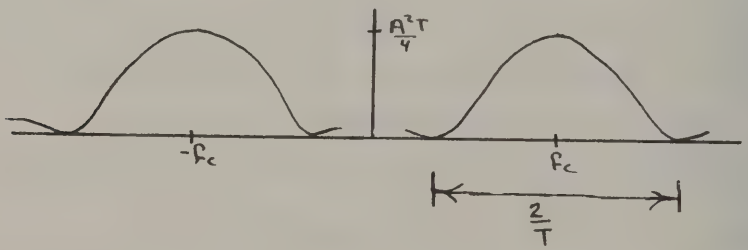
This spectrum is shown below.



(b) Note that the power spectrum for binary PSK with bit period T is

$$S_{x+}(f) = \frac{A^2}{4}T\left[\operatorname{sinc}^2 T(f-f_c) + \operatorname{sinc}^2 T(f+f_c)\right]$$

and is shown below.



Thus the bandwidth of QPSK is ½ that of PSK for equal data rates.

(c) The data rate of QPSK is twice that of PSK.

(7.18)

(a) For PRK, $P_e = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{A^2T}{2N_0}}\right]$

For QPSK, $P_{e_i} = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{A^2T}{2N_0}}\right]$.

For QPSK, $\frac{1}{2}A^2T$ represents energy per quad. channel.

For PRK, T must be ½ that for QPSK.

∴ Performance is the same.

(b) On the basis of remarks in (a), we see that PRK requires 3dB more energy to achieve same error performance.

(c) Bandwidth, system complexity.

(7.19)

Exact $P_e = 1-(1-P_{e_1})^2 = 2P_{e_1} - P_{e_1}^2$

Approximate $P_e = 2P_{e_1}$.

∴ Error term is $P_{e_1}^2 = \left[\frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{A^2T}{2N_0}}\right)\right]^2$.

Using approximation for erfc we have

$$\text{Error} = \epsilon = \left[\frac{1}{2u\sqrt{\pi}}e^{-u^2}\right]^2 \quad \text{where } u = \sqrt{\frac{A^2T}{2N_0}}$$

$$= \frac{1}{4\pi u^2}e^{-2u^2} = \frac{1}{4\pi E/N_0}e^{-2E/N_0}.$$

Thus,

$E/N_0 = 0$ dB	$\Rightarrow$	$\epsilon = 0.011$	$(P_e = 1.99\times10^{-1})$
$E/N_0 = 3$ dB	$\Rightarrow$	$\epsilon = 7.29\times10^{-4}$	$(P_e = 5.33\times10^{-2})$
$E/N_0 = 6$ dB	$\Rightarrow$	$\epsilon = 6.67\times10^{-6}$	$(P_e = 5.16\times10^{-3})$
$E/N_0 = 9$ dB	$\Rightarrow$	$\epsilon = 1.12\times10^{-9}$	$(P_e = 6.69\times10^{-5})$

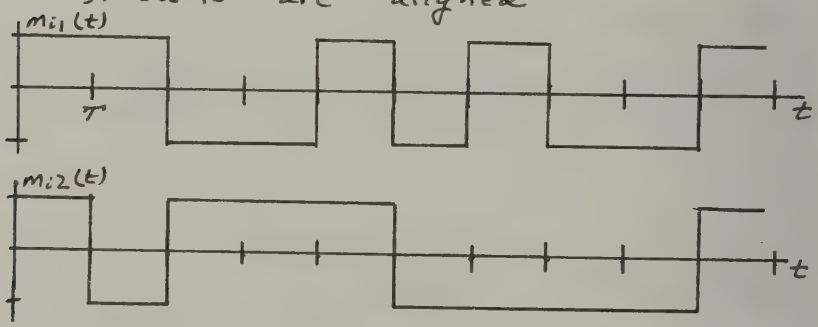
7.20

$$\{d(t)\} = \{1\,1\,1\,0\,0\,1\,0\,1\,1\,1\,0\,0\,1\,0\,0\,0\,0\,0\,1\,1\}$$

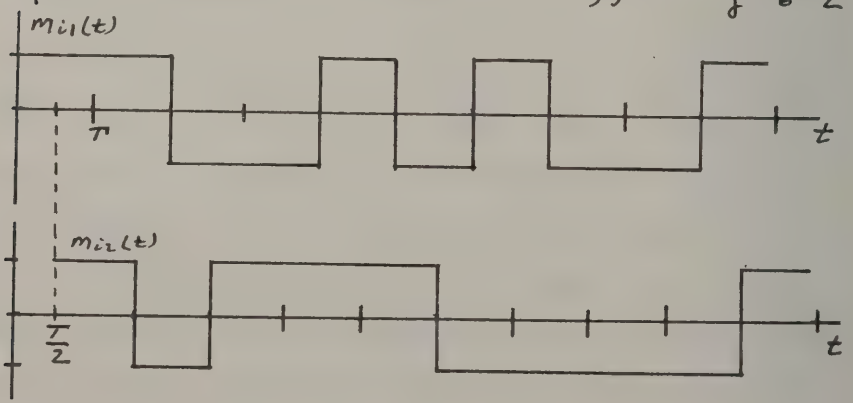
is the serial data sequence. These are demultiplexed into the two parallel data sequences

$$\{m_{i1}\} = \{1\,1\,0\,0\,1\,0\,1\,0\,0\,1\}$$
$$\{m_{i2}\} = \{1\,0\,1\,1\,1\,0\,0\,0\,0\,1\}$$

(a) For QPSK these two data streams are aligned



Multiply the above data streams by $\cos \omega_0 t$ and $\sin \omega_0 t$ to get the quadrature signals which are added to produce the QPSK modulated signal. (b) For OQPSK the parallel data streams are offset by $T_b = \frac{T}{2}$



(7.20) (cont)

These are again multiplied by $\cos\omega_0 t$ and $\sin\omega_0 t$ to produce the modulated OQPSK signal when added.

(c) MSK-I : Multiply OQPSK waveforms by $\sin(\omega t/T)$ and $\cos(\omega t/T)$

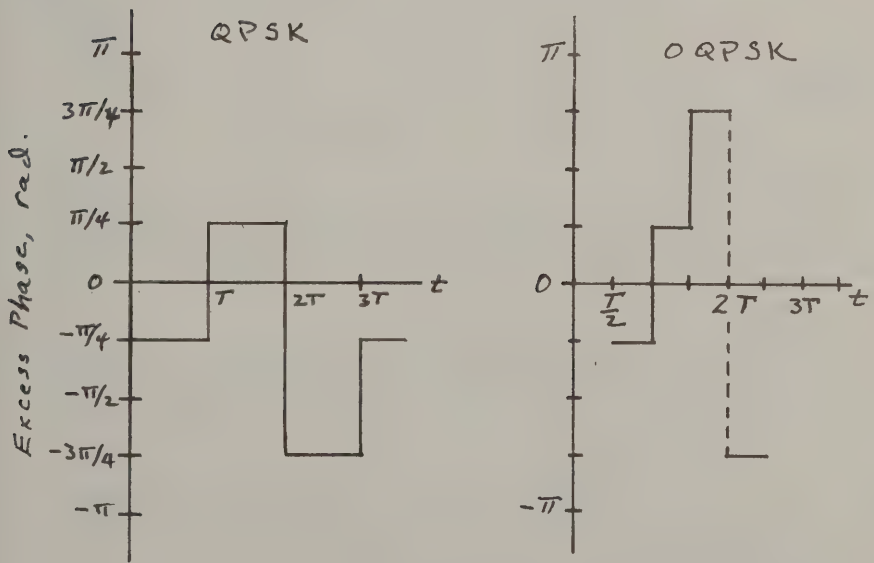
(d) MSK-II : Multiply OQPSK waveforms by $|\sin(\omega t/T)|$ and $|\cos(\omega t/T)|$

(7.21) Trellis diagrams are sketched by noting that instantaneous excess phase is

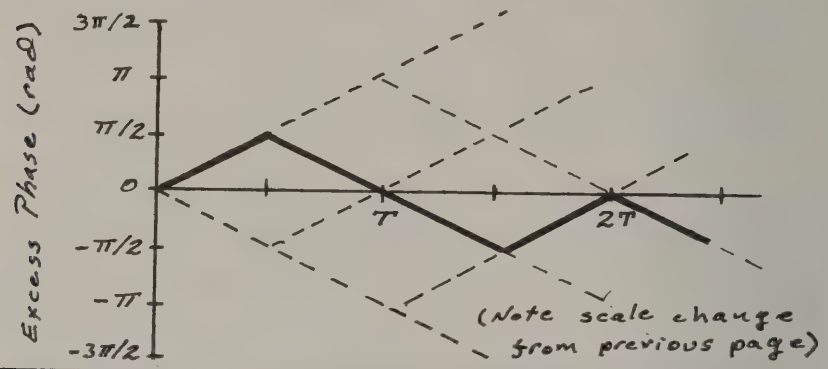
$$\theta_i(t) = -\tan^{-1}\left[\frac{m_{iz}(t)}{m_{i1}(t)}\right] \quad \text{QPSK \& OQPSK}$$

$$\theta_i(t) = -\tan^{-1}\left[a_{iz}(t)\tan\omega_1 t / a_{i1}(t)\right] \quad \text{MSK}$$

$$= \pm\frac{1}{2}\frac{2\pi}{\lambda T}t + u_k, \quad u_k = 0 \text{ or } \pi \text{ mod } 2\pi$$



(7.21) (Cont) MSK - phase trellis



(7.22) For simplicity, use single-sided spectra

Conversion filter: $H_c(f) = \text{sinc}\left[(f-f_c)T_b - 0.25\right]$

PRK Spectrum: $S'_{PRK}(f) = \frac{A^2T_b}{2}\,\text{sinc}^2\left[(f-f_c)T_b + 0.25\right]$

$$S'_{MSK}(f) = \frac{A^2T_b}{2}\,\text{sinc}^2\left[(f-f_c)T_b + 0.25\right]\text{sinc}^2\left[(f-f_c)T_b - 0.25\right]$$

$$= \frac{A^2T_b}{2}\,\frac{\sin^2\left[\pi(f-f_c)T_b + \frac{\pi}{4}\right]\sin^2\left[\pi(f-f_c)T_b - \frac{\pi}{4}\right]}{\left[\pi(f-f_c)T_b + \frac{\pi}{4}\right]^2\left[\pi(f-f_c)T_b - \frac{\pi}{4}\right]^2}$$

Since $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

We have

$$S'_{MSK}(f) = \frac{A^2T_b\left[\sin\alpha + \cos\alpha\right]^2\left[\sin\alpha - \cos\alpha\right]^2}{2(4)\pi^4\left[(f-f_c)T_b + \frac{1}{4}\right]^2\left[(f-f_c)T_b - \frac{1}{4}\right]^2}$$

where $\alpha = \pi(f-f_c)T_b$

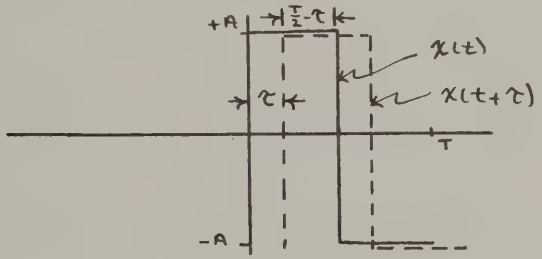
$$S_{MSK}(f) = \frac{2A^2T_b}{\pi^4}\,\frac{\left\{\sin^2\left[\pi(f-f_c)T_b\right] - \cos^2\left[\pi(f-f_c)T_b\right]\right\}^2}{\left[4(f-f_c)T_b + 1\right]^2\left[4(f-f_c)T_b - 1\right]^2}$$

$$= \frac{2A^2T_b}{\pi^4}\,\frac{\cos^2\left[2\pi(f-f_c)T_b\right]}{\left[1 - 16(f-f_c)^2T_b^2\right]^2}$$

(7.23)

Note that the solution is easier to obtain using ensemble averaging and so this is the technique used here.

We know that $R(\tau) = 0$, $|\tau| \geq T$ so we need only consider $|\tau| \leq T$. Also, from symmetry we need only consider $0 \leq \tau \leq T$. One period of a typical waveform and a shifted version is shown below.



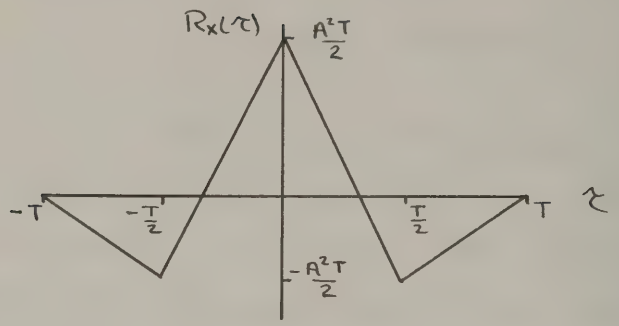
Without loss of generality, we assume $x(t) = 1$, $0 \leq t \leq T/2$. Furthermore, we need only consider that portion of the waveform shown since other contributions are zero with $P[x(t)=1] = P[x(t)=-1] = \frac{1}{2}$. Thus

$$E\{x(t)\,x(t+\tau)\} = \begin{cases} A^2\left[\left(\frac{T}{2}-\tau\right)-\tau+\left(\frac{T}{2}-\tau\right)\right] & \tau \leq \frac{T}{2} \\ -A^2(T-\tau) & \frac{T}{2} \leq \tau \leq T. \end{cases}$$

$$= \begin{cases} A^2[T-3\tau] & |\tau| \leq T/2 \\ -A^2[T-\tau] & \frac{T}{2} \leq |\tau| \leq T. \\ 0 & |\tau| > T \end{cases}$$

(7.23) (Cont.)

This function is plotted below.



In terms of triangular functions this is

$$R_x(\tau) = 2A^2T\,\Lambda\left(\frac{2\tau}{T}\right) - A^2T\,\Lambda\left(\frac{\tau}{T}\right).$$

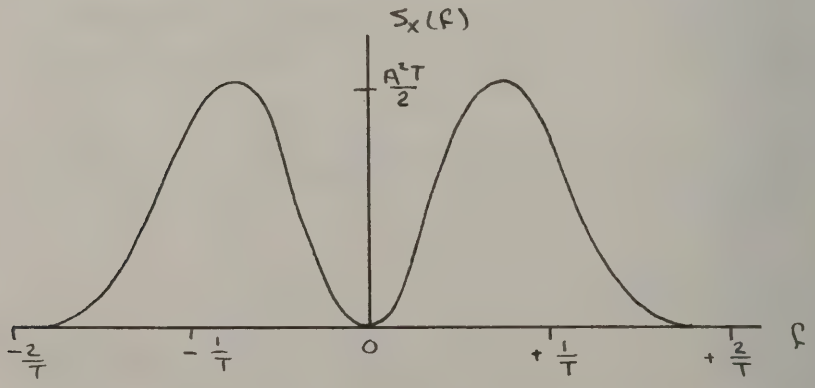
Since

$$\Lambda(\tau/T_0) \leftrightarrow \tau_0 \operatorname{sinc}^2 \tau_0 f$$

the power spectral density of $x(t)$ is

$$S_x(f) = A^2T^2 \operatorname{sinc}^2 \frac{Tf}{2} - A^2T^2 \operatorname{sinc}^2 Tf.$$

This function is plotted below.



7.24

(a) Given $d(t)$ a random sequence with $R_d(\tau) = 1 - |\frac{\tau}{T}|$, $|\tau| \le T$. Consider $y(t) = \frac{A}{2}(1+d(t))\cos(\omega_0 t + \theta)$ where θ is uniformly distributed in 0 to 2π and independent of $d(t)$.

Then,

$$E\{y(t)\,y(t+\tau)\} = E\left\{\frac{A^2}{4}[1+d(t)][1+d(t+\tau)]\cos(\omega_0 t+\theta)\cdot \cos(\omega_0 t+\theta+\omega_0\tau)\right\}$$

$$= \frac{A^2}{4} E\{[1+d(t)][1+d(t+\tau)]\}\, E\{\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\}$$

$1+R_d(\tau)$ for $E\{d(t)\}=0$. $\quad \frac{1}{2}\cos\omega_0\tau$ (see Chap. 5)

$$\therefore R_y(\tau) = \frac{A^2}{8}[1+R_d(\tau)]\cos\omega_0\tau.$$

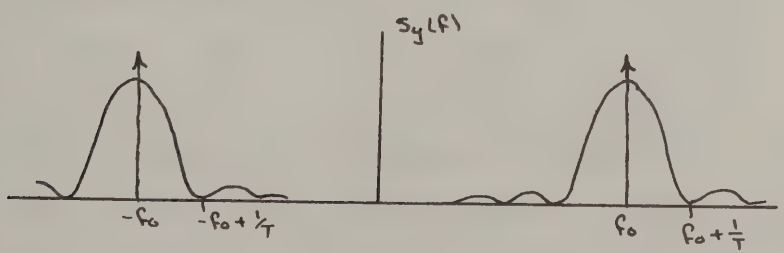
By transform theorems,

$$S_y(f) = \frac{A^2}{8}\left[\tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0) + \tfrac{1}{2}S_d(f-f_0) + \tfrac{1}{2}S_d(f+f_0)\right]$$

From Fourier Transform tables,

$$S_d(f) = \frac{1}{T\pi^2 f^2}\sin^2(\pi f T).$$

Thus, we can sketch $S_y(f)$:



(b) Consider $y(t) = A\sin[\omega_0 t + \cos^{-1} m d(t) + \theta]$ where $d(t)$ and $R_d(\tau)$ are as defined in part (a), and θ is uniform over 0 to 2π.

(7.24) (Cont.)

Let $K = \cos^{-1} m$. Then we can write

$$R_y(\tau) = E\{y(t)\,y(t+\tau)\} = E\{A^2 \sin[\omega_0 t + Kd(t) + \theta] \cdot \sin[\omega_0 t + \omega_0\tau + Kd(t+\tau) + \theta]\}$$

$$= E\left\{\frac{A^2}{2}\cos[Kd(t) - Kd(t+\tau) - \omega_0\tau]\right\} - \underbrace{E\left\{\frac{A^2}{2}\cos[2\omega_0 t + \omega_0\tau + Kd(t) + Kd(t+\tau) + 2\theta]\right\}}$$

This term is zero since $E\{\cos[f(t)+\theta]\}$ is zero $\forall\, f(t)$.

$$= E\left\{\frac{A^2}{2}\cos[Kd(t) - Kd(t+\tau) - \omega_0\tau]\right\}$$

$$= E\left\{\frac{A^2}{2}\cos[Kd(t) - Kd(t+\tau)]\cos\omega_0\tau\right\} + E\left\{\frac{A^2}{2}\sin[Kd(t) - Kd(t+\tau)]\sin\omega_0\tau\right\}$$

$$= E\left\{\frac{A^2}{2}[\cos Kd(t)\cos Kd(t+\tau) + \sin Kd(t)\sin Kd(t+\tau)]\cos\omega_0\tau\right\}$$

$$+ E\left\{\frac{A^2}{2}[\sin Kd(t)\cos Kd(t+\tau) - \cos Kd(t)\sin Kd(t+\tau)]\sin\omega_0\tau\right\}.$$

Note that since $d(t) = \pm 1$ only we can write

$$\cos[Kd(t)] = \cos K = \cos[\cos^{-1} m] = m.$$

$$\sin[Kd(t)] = d(t)\sin K = d(t)\sin[\cos^{-1} m] = d(t)\sqrt{1-m^2}.$$

$$\therefore E\{y(t)\,y(t+\tau)\} = E\left\{\frac{A^2}{2}[m^2 + d(t)\,d(t+\tau)\sqrt{1-m^2}]\cos\omega_0\tau\right\}$$

$$+ E\left\{\frac{A^2}{2}[m\sqrt{1-m^2}\,d(t) - m\sqrt{1-m^2}\,d(t+\tau)]\sin\omega_0\tau\right\}$$

$$= \frac{A^2 m^2}{2}\cos\omega_0\tau + \frac{A^2\sqrt{1-m^2}}{2}R_d(\tau)\cos\omega_0\tau.$$

Note that $E\{d(t)\} = 0$ has been used.

Thus, it follows that

$$S_y(f) = \frac{A^2 m^2}{4}[\delta(f-f_0) + \delta(f+f_0)] + \frac{A^2\sqrt{1-m^2}}{4}[S_d(f-f_0) + S_d(f+f_0)]$$

where $S_d(f)$ is as given in part (a).

For various values of m, $S_y(f)$ is

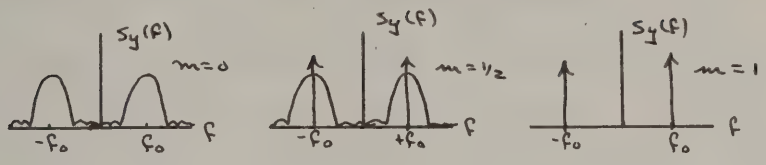
$m=0$: $S_y(f) = \frac{A^2}{4}[S_d(f-f_0) + S_d(f+f_0)]$

$m=\frac{1}{2}$: $S_y(f) = \frac{A^2}{16}[\delta(f-f_0) + \delta(f+f_0)] + \frac{A^2\sqrt{3}}{8}[S_d(f-f_0) + S_d(f+f_0)]$

$m=1$: $S_y(f) = \frac{A^2}{4}[\delta(f-f_0) + \delta(f+f_0)]$.

(7.24) (cont)

The plots of $S_y(f)$ are shown below.



(c) Combine the results of part (a).

(7.25) To show that

$$\mathcal{F}\left\{\cos\left(\frac{\pi t}{2T_b}\right)\Pi\left(\frac{t}{2T_b}\right)\right\} = \frac{4T_b}{\pi}\,\frac{\cos^2(2\pi T_b f)}{1-(4T_b f)^2}$$

First write

$$\cos\left(\frac{\pi t}{2T_b}\right)\Pi\left(\frac{t}{2T_b}\right) = \frac{1}{2}\Pi\left(\frac{t}{2T_b}\right)e^{j\pi t/2T_b} + \frac{1}{2}\Pi\left(\frac{t}{2T_b}\right)e^{-j\pi t/2T_b}$$

Use $\Pi(t/\tau) \longleftrightarrow \tau\,\mathrm{sinc}(f\tau)$ with $\tau = 2T_b$ and the modulation theorem to write

$$\cos\left(\frac{\pi t}{2T_b}\right)\Pi\left(\frac{t}{2T_b}\right) \longleftrightarrow T_b\,\mathrm{sinc}\left[2T_b(f-1/4T_b)\right] + T_b\,\mathrm{sinc}\left[2T_b(f+1/4T_b)\right]$$

Rewrite the right-hand side (RHS) as

$$\mathrm{RHS} = T_b\left[\frac{\sin\left(2\pi T_b f - \frac{\pi}{2}\right)}{2\pi T_b f - \frac{\pi}{2}} + \frac{\sin\left(2\pi T_b f + \frac{\pi}{2}\right)}{2\pi T_b f + \frac{\pi}{2}}\right]$$

Since $\sin\left(2\pi T_b f \pm \frac{\pi}{2}\right) = \pm\cos(2\pi T_b f)$

$$\mathrm{RHS} = T_b\left[-\frac{\cos(2\pi T_b f)}{2\pi T_b f - \frac{\pi}{2}} + \frac{\cos(2\pi T_b f)}{2\pi T_b f + \frac{\pi}{2}}\right]$$

$$= \frac{2T_b}{\pi}\left[\frac{1}{1-4T_b f} + \frac{1}{1+4T_b f}\right]\cos(2\pi T_b f)$$

$$= \frac{4T_b}{\pi}\,\frac{\cos(2\pi T_b f)}{1-(4T_b f)^2}$$

(7.26)

$$x_c(t) = A[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t]$$
$$= A\,\mathrm{Re}\left\{[m_1(t) - j m_2(t)]\exp(j\omega_c t)\right\}$$

Therefore consider the complex baseband signal $m(t) = [m_1(t) - j m_2(t)]A$

Let $G(f)$ = PSD of $m(t)$ or, from (5.6)

$$G(f) = \lim_{T'\to\infty} \frac{E\{|\mathcal{F}[m_{T'}(t)]|^2\}}{T'}$$ where T'

is a long time interval centered at $t=0$ and $m_{T'}(t) = m(t)\,\Pi(t/T')$. Let $T' = (2K+1)T$ so that T' is an integer multiple of T. Also let

$$m_{T'}(t) = m_{1T'}(t) - j m_{2T'}(t)$$
$$= \sum_{k=-K}^{K}\left[a_k p(t - kT - \Delta_1) - j b_k q(t - kT - \Delta_2)\right]$$

see (7.116) and (7.117)

$$\mathcal{F}[m_{T'}(t)] = \sum_{k=-K}^{K}\left\{a_k \mathcal{F}[p(t-kT-\Delta_1)] - j b_k \mathcal{F}[q(t-kT-\Delta_2)]\right\}$$

But $\mathcal{F}[p(t-kT-\Delta_1)] = P(f)\,e^{-j2\pi f(kT+\Delta_1)}$

$\mathcal{F}[q(t-kT-\Delta_2)] = Q(f)\,e^{-j2\pi f(kT+\Delta_2)}$

where $P(f) = \mathcal{F}[p(t)]$ and $Q(f) = \mathcal{F}[q(t)]$

Thus $\mathcal{F}[m_{T'}(t)]$

$$= \sum_{k=-K}^{K}\left\{a_k P(f) e^{-j2\pi f(kT+\Delta_1)} + b_k Q(f) e^{-j2\pi f(kT+\Delta_2)}\right\}$$

Therefore $\overline{|\mathcal{F}[m_{T'}(t)]|}$

$$= \sum_{k,l=-K}^{K} \overline{\left[a_k P(f) e^{-j2\pi f(kT+\Delta_1)} - j b_k Q(f) e^{-j2\pi f(kT+\Delta_2)}\right]} \cdots$$
$$\overline{\times\left[a_l P(f) e^{+j2\pi f(lT+\Delta_1)} + j b_l Q(f) e^{+j2\pi f(lT+\Delta_2)}\right]}$$

where the overbar denotes the average over a_k, b_k, Δ_1, and Δ_2. But $\{a_k\}$, $\{b_k\}$, Δ_1 and Δ_2

(7.26) (Cont.)

are independent with $\overline{a_k a_\ell} = \begin{cases} A, & k=\ell \\ 0, & k\neq\ell \end{cases}$ and $\overline{b_k b_\ell} = \begin{cases} B^2, & k=\ell \\ 0, & k\neq\ell \end{cases}$. Therefore, the average of the double sum can be written

$$\overline{|\mathcal{F}[m_T(t)]|}$$

$$= \sum_{k,\ell=-K}^{K} \sum \left\{ \overline{a_k a_\ell} |P(f)|^2 e^{-j2\pi f(k-\ell)T} + \overline{b_k b_\ell} |Q(f)|^2 e^{-j2\pi f(k-\ell)T} \right\}$$

$$= \sum_{k,\ell=-K}^{K} \sum \left\{ A^2 \delta_{k\ell} |P(f)|^2 e^{-j2\pi f(k-\ell)T} + B^2 \delta_{k\ell} |Q(f)|^2 e^{-j2\pi f(k-\ell)T} \right\}$$

$$= \sum_{k=-K}^{K} \left\{ A^2 |P(f)|^2 + B^2 |Q(f)|^2 \right\}$$

$$= (2K+1)\left(A^2 |P(f)|^2 + B^2 |Q(f)|^2\right)$$

Note that in the above $\delta_{k\ell} = \begin{cases} 1, & k=\ell \\ 0, & k\neq\ell \end{cases}$

$$G(f) = \lim_{K\to\infty} \frac{(2K+1)\left[A^2 |P(f)|^2 + B^2 |Q(f)|^2\right]}{(2K+1)T + \underbrace{2\varepsilon}_{\text{due to end of interval}}}$$

Thus,

$$G(f) = \frac{1}{T}\left\{A^2 |P(f)|^2 + B^2 |Q(f)|^2\right\}$$

(7.27)

The required expressions are

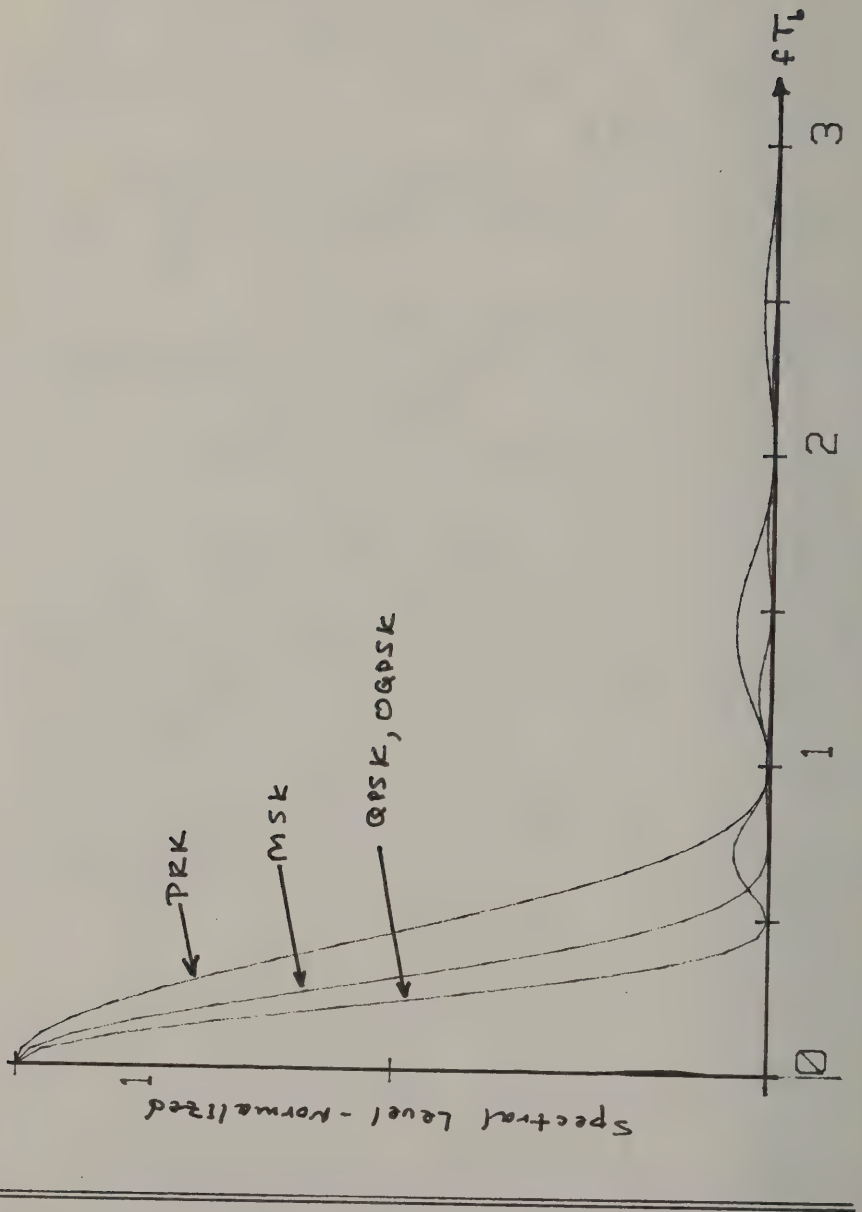
$$G_{PRK}(f) = A^2 T_b \, \text{sinc}^2(T_b f) \qquad (7.120)$$

$$G_{QPSK}(f) = A^2 T_b \, \text{sinc}^2(2T_b f) \qquad (7.121)$$

$$G_{MSK}(f) = \frac{16A^2 T_b}{\pi^2} \frac{\cos^2(2\pi T_b f)}{\left[1-(4T_b f)^2\right]^2} \qquad (7.124)$$

All plots (shown on the following page) have been normalized to a maximum value of one.

7.27 (cont)



(7.28)

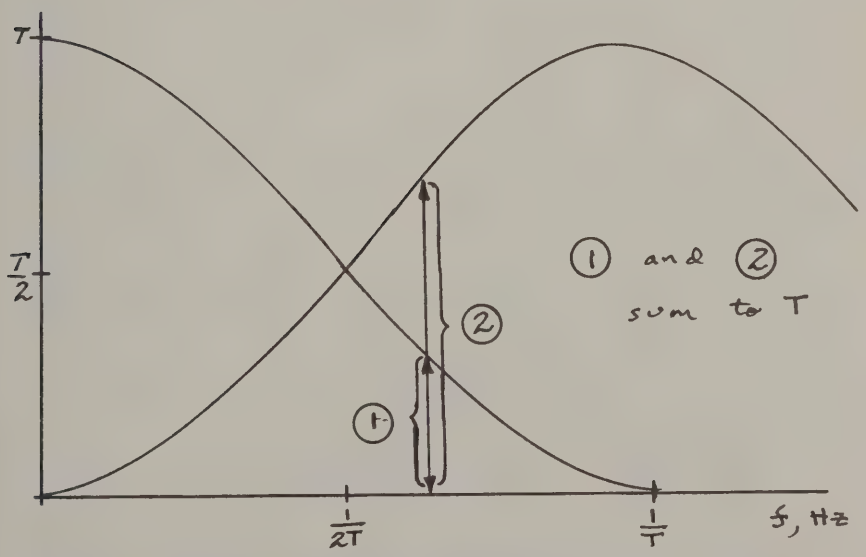
Use Plots of Prob. 7.27 and Fig. 7.32

Modulation Type	Null-to-Null BW	90% BW (-10 dB)	99% BW (-20 dB)
PRK	$2/T_b$	$1.6/T_b$	$14/T_b$
QPSK, OQPSK	$1/T_b$	$0.8/T_b$	$7/T_b$
MSK	$3/T_b$	$0.8/T_b$	$1.2/T_b$

Ratios in terms of BW:

Criterion	Most Efficient
Null-to-Null	QPSK
90% Power Containment	QPSK or MSK
99% Power Containment	MSK

(7.29) An example is shown below for $\beta = \frac{1}{2T}$



(7.30)

$$\sigma^2 = \text{var}(N)$$

where $N = n(t) * h_R(t)\big|_{t=t_0}$

$$= \int_{-\infty}^{\infty} n(\lambda)\, h_R(t_0 - \lambda)\, d\lambda$$

In terms of the PSD of $n(t)$, $G_n(f)$,

$$\sigma^2 = \int_{-\infty}^{\infty} G_n(f)\,|H_R(f)|^2\, df$$

Now express A^2 in terms of the transmitted signal energy: From (7.134) the transmitted signal is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, h_T(t - kT)$$

From Problem 5.19, the PSD of this signal is

$$\mathcal{S}_x(f) = \frac{1}{T}\,\text{var}(a_k)\,|H_T(f)|^2, \quad \text{var}(a_k) = 1$$

where ν in Prob. 5.19 is $\frac{1}{T}$ here

The energy per pulse is

$$E_T = T P_{ave} = T\int_{-\infty}^{\infty} \mathcal{S}_x(f)\, df = \int_{-\infty}^{\infty} |H_T(f)|^2 df$$

Use (7.138) to put this in terms of A^2.

$$|H_T(f)|^2 = A^2 |P(f)|^2 / |H_c(f)|^2 |H_R(f)|^2$$

Thus

$$E_T = A^2 \int_{-\infty}^{\infty} \frac{|P(f)|^2}{|H_c(f)|^2 |H_R(f)|^2}\, df$$

or

$$A^2 = E_T \Big/ \int_{-\infty}^{\infty} \frac{|P(f)|^2}{|H_c(f)|^2 |H_R(f)|^2}\, df$$

$$\frac{\sigma^2}{A^2} = \frac{1}{E_T}\int_{-\infty}^{\infty} G_n(f)\,|H_R(f)|^2 df \int_{-\infty}^{\infty} \frac{|P(f)|^2\, df}{|H_c(f)|^2 |H_R(f)|^2}$$

(7.30) (Cont.)

In Schwarz's inequality

$$\left|\int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df\right|^2 \le \int_{-\infty}^{\infty}|X(f)|^2\,df\int_{-\infty}^{\infty}|Y(f)|^2\,df$$

Let $|X(f)| = G_n^{1/2}(f)\,|H_R(f)|$

and let $|Y(f)| = |P_r(f)| / |H_c(f)|\,|H_R(f)|$

we then have

$$\text{(*)}\quad \frac{\sigma^2}{A^2} \ge \frac{1}{E_T}\left[\int_{-\infty}^{\infty}\frac{G_n^{1/2}(f)\,|P_r(f)|}{|H_c(f)|}\,df\right]^2$$

with equality if and only if

$$|X(f)| = K^2\,|Y(f)|$$

or

$$G_n^{1/2}(f)\,|H_R(f)|_{opt} = K^2\frac{|P_r(f)|}{|H_c(f)|\,|H_R(f)|_{opt}}$$

or

$$|H_R(f)|_{opt}^2 = K^2\frac{|P_r(f)|}{G_n^{1/2}(f)\,|H_c(f)|}$$

or

$$|H_R(f)|_{opt} = K\frac{|P_r(f)|^{1/2}}{G_n^{1/4}(f)\,|H_c(f)|^{1/2}}$$

where K is an arbitrary constant

From (7.138):

$$|H_T(f)|_{opt} = k'\frac{|P_r(f)|^{1/2}\,G_n^{1/4}(f)}{|H_c(f)|^{1/2}}$$

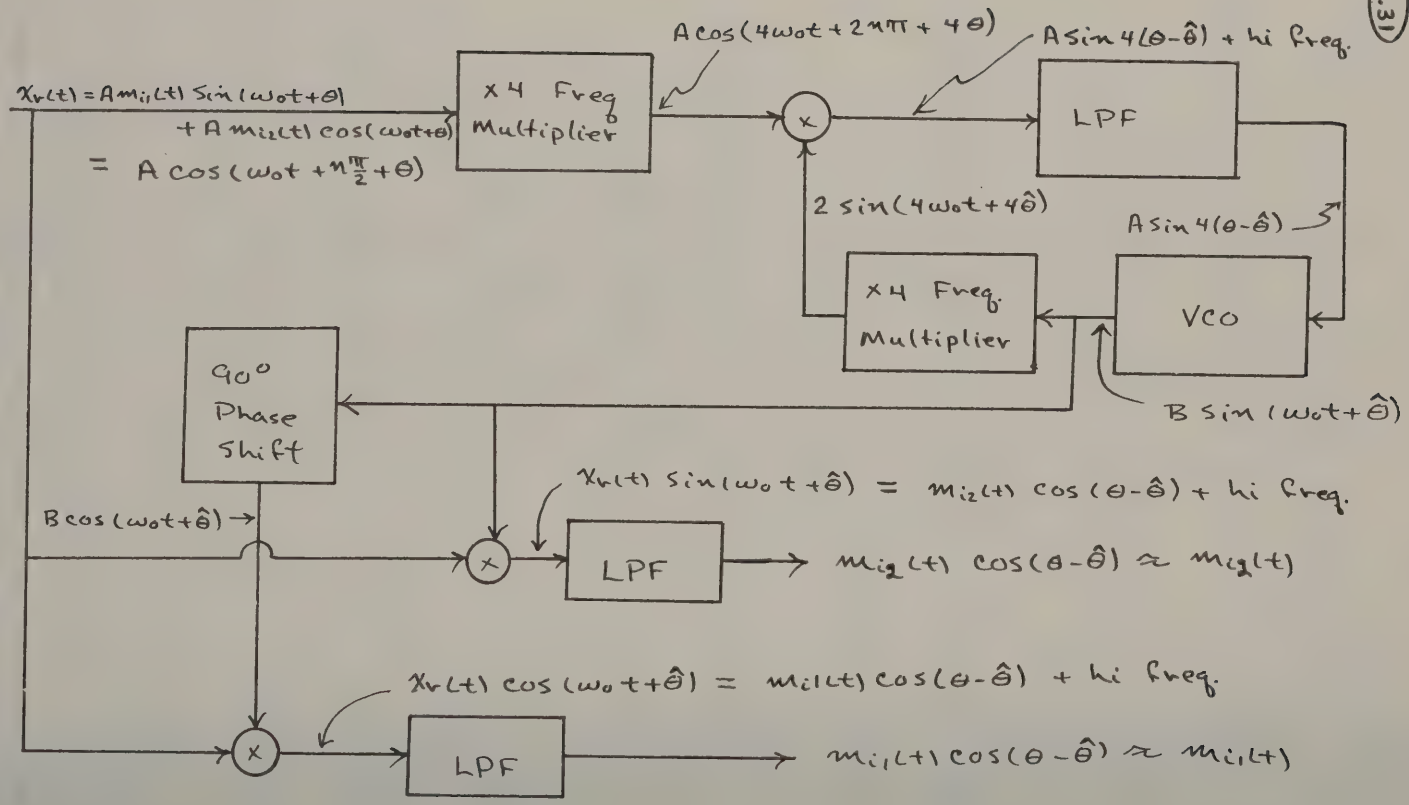
From (*)

$$\left(\frac{\sigma^2}{A^2}\right)_{min} = \frac{1}{E_T}\left[\int_{-\infty}^{\infty}\frac{G_n^{1/2}(f)\,|P_r(f)|}{|H_c(f)|}\,df\right]^2$$

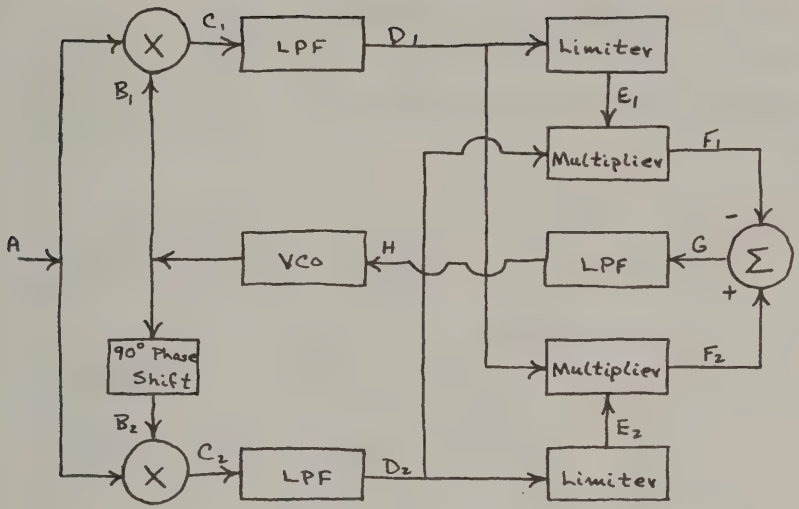
Therefore:

$$P_{min}(E) = Q\left[\sqrt{E_T}\left[\int_{-\infty}^{\infty}\frac{G_n^{1/2}(f)\,|P_r(f)|}{|H_c(f)|}\,df\right]^{-1}\right]$$

(7.31)


$x_r(t) = A m_{i1}(t) \sin(\omega_0 t + \theta)$
$+ A m_{i2}(t) \cos(\omega_0 t + \theta)$
$= A \cos(\omega_0 t + n\frac{\pi}{2} + \theta)$
× 4 Freq Multiplier
$A \cos(4\omega_0 t + 2n\pi + 4\theta)$
×
$A \sin 4(\theta - \hat{\theta})$ + hi freq.
LPF
$2 \sin(4\omega_0 t + 4\hat{\theta})$
$A \sin 4(\theta - \hat{\theta})$
× 4 Freq. Multiplier
VCO
$B \sin(\omega_0 t + \hat{\theta})$
90° Phase Shift
$B \cos(\omega_0 t + \hat{\theta})$
$x_r(t) \sin(\omega_0 t + \hat{\theta}) = m_{i2}(t) \cos(\theta - \hat{\theta})$ + hi freq.
×
LPF
$m_{i2}(t) \cos(\theta - \hat{\theta}) \approx m_{i2}(t)$
$x_r(t) \cos(\omega_0 t + \hat{\theta}) = m_{i1}(t) \cos(\theta - \hat{\theta})$ + hi freq.
×
LPF
$m_{i1}(t) \cos(\theta - \hat{\theta}) \approx m_{i1}(t)$

(7.32)



In the following, the i subscript is suppressed.

A: Input is $x(t) = m_1(t)\sin(\omega_0 t+\theta) + m_2(t)\cos(\omega_0 t+\theta)$

B_1, B_2: Let initial VCO output be $2\sin(\omega_0 t+\hat{\theta})$.

Then $x_{B_1}(t) = 2\sin(\omega_0 t+\hat{\theta})$, $x_{B_2}(t) = 2\cos(\omega_0 t+\hat{\theta})$.

C_1: $x_{C_1}(t) = x(t)x_{B_1}(t) = m_1(t)[\cos(\theta-\hat{\theta}) - \cos(2\omega_0 t+\theta+\hat{\theta})]$

$\quad + m_2(t)[\sin(\theta-\hat{\theta}) + \sin(2\omega_0 t+\theta+\hat{\theta})]$

C_2: $x_{C_2}(t) = x(t)x_{B_2}(t) = m_1(t)[\sin(\hat{\theta}-\theta) + \sin(2\omega_0 t+\theta+\hat{\theta})]$

$\quad + m_2(t)[\cos(\theta-\hat{\theta}) + \cos(2\omega_0 t+\theta+\hat{\theta})]$

D_1: $x_{D_1}(t) = LP\{x_{C_1}(t)\} = m_1(t)\cos(\theta-\hat{\theta}) + m_2(t)\sin(\theta-\hat{\theta})$

D_2: $x_{D_2}(t) = LP\{x_{C_2}(t)\} = m_1(t)\sin(\hat{\theta}-\theta) + m_2(t)\cos(\theta-\hat{\theta})$.

E_1, E_2: Note that $|\cos(\theta-\hat{\theta})| >> |\sin(\theta-\hat{\theta})|$ so

$x_{e_1}(t) \approx m_1(t)$, $x_{e_2}(t) \approx m_2(t)$

F_1: $x_{F_1}(t) = x_{e_1}(t)x_{D_2}(t) = \cancel{m_1^2(t)}^{1.0}\sin(\hat{\theta}-\theta) + m_1(t)m_2(t)\cos(\theta-\hat{\theta})$

F_2: $x_{F_2}(t) = x_{e_2}(t)x_{D_1}(t) = m_1(t)m_2(t)\cos(\theta-\hat{\theta}) + \cancel{m_2^2(t)}^{1.0}\sin(\theta-\hat{\theta})$

(7.32) (Cont.)

G: $x_G(t) = x_{F_2}(t) - x_{F_1}(t) = 2\sin(\theta - \hat{\theta})$

H: $x_H(t) = LP\{x_G(t)\} = LP\{2\sin(\theta - \hat{\theta})\}$.

(7.33)

Since σ_θ^2 is inversely proportional to Q for all cases, we need only consider $Q=1$, then other values may be obtained by scaling.

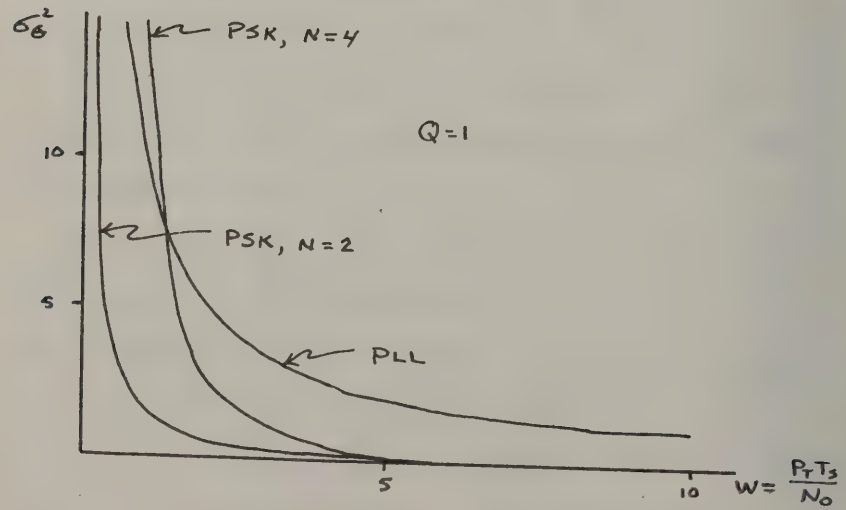
For PLL: $\sigma_\theta^2 \propto \frac{1}{z}$ where $z = \frac{P_c T_s}{N_o} = \frac{0.1\, P_T T_s}{N_o} \triangleq 0.1\,W$

where P_c = carrier power, P_T = trans. power.

$\therefore \sigma_\theta^2 = 10/W$.

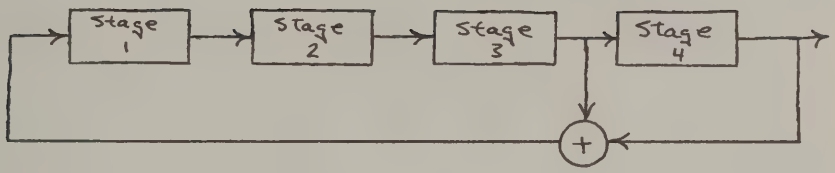
For PSK, $N=2$: $\sigma_\theta^2 = \left[\frac{1}{W} + \frac{1}{2W^2}\right]$.

For PSK, $N=4$: $\sigma_\theta^2 = \left[\frac{1}{W} + \frac{9}{2}\frac{1}{W^2} + \frac{6}{W^3} + \frac{3}{2}\frac{1}{W^4}\right]$.



Note that for small $\frac{P_T T_s}{N_o}$ that the PLL performs best, PSK with $N=2$ next, PSK with $N=4$ worst. To see this form ratios $\frac{\sigma_{\theta,PLL}^2}{\sigma_{\theta,PSK,2}^2}$, $\frac{\sigma_{\theta,PSK_2}^2}{\sigma_{\theta,PSK,4}^2}$, with $W \to 0$.

7.34

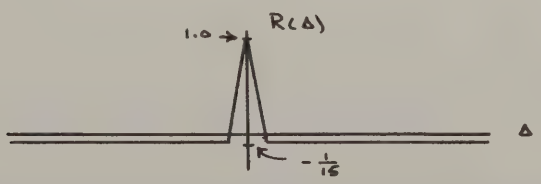


States:	Stage 1	Stage 2	Stage 3	Stage 4
T=0	1	1	1	1
1	0	1	1	1
2	0	0	1	1
3	0	0	0	1
4	1	0	0	0
5	0	1	0	0
6	0	0	1	0
7	1	0	0	1
8	1	1	0	0
9	0	1	1	0
10	1	0	1	1
11	0	1	0	1
12	1	0	1	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

Thus the output sequence is 111100010011010. The autocorrelation function can be obtained by writing the sequence on two pieces of paper and shifting one past the other. Then

$$R(\Delta) = [N_{\text{ALIKE}} - N_{\text{DIFFERENT}}] / \text{Length}.$$

The result is shown below.



(7.35)

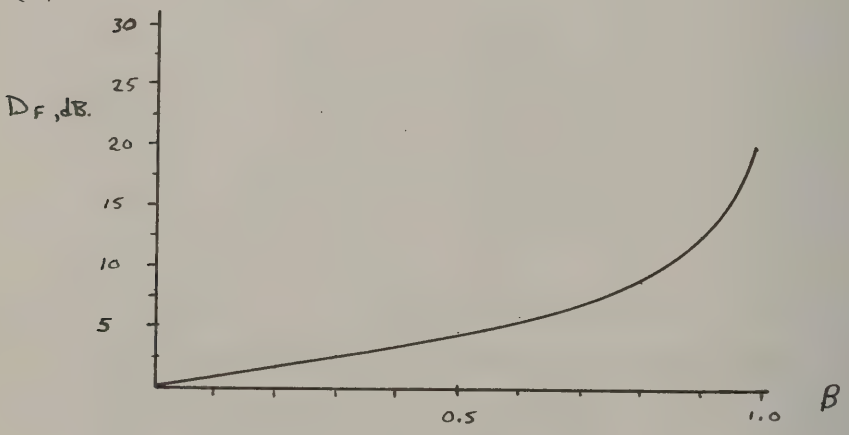
To obtain the aperiodic autocorrelation function, we slide the PN sequence past itself and compute the number of alike bits minus number of unlike bits for the overlap. The result can be scaled by length if desired.

(a) $-1\ 0\ -1\ 0\ -1\ 0\ 7\ 0\ -1\ 0\ -1\ 0\ -1$, $|MAX| = 1$.

(b) $-1\ 0\ -1\ 0\ 3\ 0\ 1\ -2\ -1\ -4\ -1\ 0\ -1\ 0\ 15\ \ldots$, $|MAX| = 4$.

(7.36)

(a)



(b) w/o Fading: $P_e = 10^{-4} = \frac{1}{2\sqrt{\pi z}} e^{-z} \Rightarrow z = 9\text{dB}$.

From curve of part (a): $D_F = 6\text{ dB} \Rightarrow \beta = 0.3$

From plot, Fig. 7.43: $\beta = 0.3$, $P_e = 10^{-4} \Rightarrow Z_0 = 11\text{dB}$

$\therefore$ Increase in SNR $\approx$ 2 dB

(c) w/o Fading: $P_e = \frac{1}{2\sqrt{\pi z}} e^{-z}$, $z = 10\text{dB} \Rightarrow P_e = 4 \times 10^{-6}$

with Fading (from Fig. 7.43): $P_e = 2 \times 10^{-4}$.

(7.37)

$$P_e = \tfrac{1}{4}\operatorname{erfc}\left[\sqrt{z_0(1+\delta)}\right] + \tfrac{1}{4}\operatorname{erfc}\left[\sqrt{z_0[(1+\delta) - 2\delta\tau_m/T]}\right]$$

Letting $z_m = \frac{E}{N_0}(1+\delta) = z_0(1+\delta)$ yields

$$P_e = \tfrac{1}{4}\operatorname{erfc}\left[\sqrt{z_m}\sqrt{1+\delta}\right] + \tfrac{1}{4}\operatorname{erfc}\left\{\sqrt{z_m}\left[\sqrt{1+\delta} - \frac{2\delta\tau_m}{T\sqrt{1+\delta}}\right]\right\}$$

$$= \tfrac{1}{4}\operatorname{erfc}\left[\sqrt{z_m(1+\delta)}\right] + \tfrac{1}{4}\operatorname{erfc}\left\{\sqrt{z_m\left[\sqrt{1+\delta} - \frac{2\delta\tau_m}{T\sqrt{1+\delta}}\right]^2}\right\}$$

In Fig 7.5 $P_e = \frac{1}{2}\operatorname{erfc}(\sqrt{z})$ is plotted. To use this figure, scale the magnitude by $\frac{1}{2}$ and shift by $10\log[1+\delta]$ or $20\log\left[\sqrt{1+\delta} - \frac{2\delta\tau_m}{T\sqrt{1+\delta}}\right]$. Then combine the terms to find P_e. Thus

$\tau_m/T = 0.2$: $P_e = \frac{1}{2}P_e\{z_m + 1.76\} + \frac{1}{2}P_e\{z_m + 0.166\}$

$\tau_m/T = 0.6$: $P_e = \frac{1}{2}P_e\{z_m + 1.76\} + \frac{1}{2}P_e\{z_m + 0.540\}$

$\tau_m/T = 1.0$: $P_e = \frac{1}{2}P_e\{z_m + 1.76\} + \frac{1}{2}P_e\{z_m + 0.166\}$

(7.38) From (7.176)

$$[P_c] = \begin{bmatrix} 1.0 & 0.1 & -0.01 \\ 0.2 & 1.0 & 0.1 \\ -0.02 & 0.2 & 1.0 \end{bmatrix}$$

The inverse $[P_c]^{-1}$ is found to be

$$[P_c]^{-1} = \begin{bmatrix} 1.02 & -0.11 & 0.02 \\ -0.21 & 1.04 & -0.11 \\ 0.06 & -0.21 & 1.02 \end{bmatrix}$$

The optimum weights correspond to the middle column.

(a) $P_{eq}(mT) = -0.11\,p_c[(m+1)T] + 1.04\,p_c[mT] - 0.21\,p_c[(m-1)T]$

(b) $P_{eq}(-2T) = -0.11\,p_c(-T) + 1.04\,p_c(-2T) - 0.21\,p_c(-3T)$

$$= -0.11(0.1) + 1.04(-0.01) - 0.21(0.001) = -0.02$$

(7.38) (Cont)

$$\begin{aligned} P_{eq}(-T) &= -0.11\, \rho_c(0) + 1.04\, \rho_c(-T) - 0.21\, \rho_c(-2T) \\ &= -0.11(1) + 1.04(0.1) - 0.21(-0.01) \\ &= 3.9 \times 10^{-3} \quad \text{(should be zero. this is close enough)} \end{aligned}$$

$$\begin{aligned} P_{eq}(0) &= -0.11(0.2) + 1.04(1.0) - 0.21(0.1) \\ &= 1.0 \quad \text{(as it should be)} \end{aligned}$$

(7.39)

(a) $y(t) = A\,\Delta(t) + bA\,\Delta(t-T) + n(t)$
where $T_m = T$ has been assumed

$$\begin{aligned} R_{yy}(\lambda) &= E\{y(t)\,y(t+\lambda)\} \\ &= E\{[A\Delta(t) + bA\Delta(t-T) + n(t)] \\ &\quad \times [A\Delta(t+\lambda) + bA\Delta(t+\lambda-T) + n(t+\lambda)]\} \\ &= (1+b^2)A^2 R_{\Delta\Delta}(\lambda) \\ &\quad + bA^2[R_{\Delta\Delta}(\lambda-T) + R_{\Delta\Delta}(\lambda+T)] \\ &\quad + R_n(\lambda) \end{aligned}$$

where $R_{\Delta\Delta}(\lambda) = E\{\Delta(t)\Delta(t+\lambda)\} = \Lambda(\lambda/T)$
$R_{nn}(\lambda)$ = autocorrelation of the noise $= (\pi N_0 f_3/2)\, e^{-2\pi f_3|\lambda|}$
$= (\pi N_0/2T)\exp(-2\pi|\lambda|/T)$
in which $f_3 = 1/T$ has been assumed
We need to evaluate $R_{yy}(\lambda)$ at multiples of $\Delta = T$

$$R_{yy}(0) = (1+b^2)A^2 + R_n(0) = (1+b^2)A^2 + \left(\frac{\pi N_0}{2T}\right)$$

$$R_{yy}(T) = bA^2 + R_n(T) = bA^2 + \left(\frac{\pi N_0}{2T}\right) e^{-2\pi}$$

$$R_{yy}(2T) = R_n(2T) = \left(\frac{\pi N_0}{2T}\right) e^{-4\pi}$$

Also note that $R_{yy}(\lambda) = R_{yy}(-\lambda)$

7.39 (cont)

$$\frac{T}{N_0}[R_{YY}] = \begin{bmatrix} (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & \frac{\pi}{2}e^{-4\pi} \\ b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} \\ \frac{\pi}{2}e^{-4\pi} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} \end{bmatrix}$$

(b) Need $R_{Y\alpha}(\lambda) = E\{y(t)\,\alpha(t+\lambda)\}$

$= E\{(A\alpha(t) + b A\alpha(t-T) + n(t))\,\alpha(t+\lambda)\}$

$= A\,R_{\alpha\alpha}(\lambda) + bA\,R_{\alpha\alpha}(\lambda+T)$

$$[R_{y\alpha}] = \begin{bmatrix} R_{y\alpha}(-T) \\ R_{y\alpha}(0) \\ R_{y\alpha}(T) \end{bmatrix} \Rightarrow \begin{matrix} R_{y\alpha}(-T) = bA = R_{y\alpha}(T) \\ R_{y\alpha}(0) = A \end{matrix}$$

Thus, the equations for the optimum weights are

$$\begin{bmatrix} (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & \frac{\pi}{2}e^{-4\pi} \\ b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} \\ \frac{\pi}{2}e^{-4\pi} & b\frac{E_b}{N_0} + \frac{\pi}{2}e^{-2\pi} & (1+b^2)\frac{E_b}{N_0} + \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} A\,a_{-1} \\ A\,a_0 \\ A\,a_1 \end{bmatrix}$$

$$= \begin{bmatrix} b\,E_b/N_0 \\ b\,E_b/N_0 \\ b\,E_b/N_0 \end{bmatrix}$$

Let $E_b/N_0 = 10$ (10 dB) and $b = 0.1$. The matrix equation becomes

7.39 Cont.

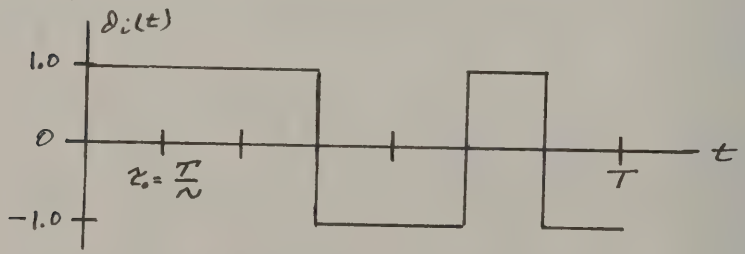
$$\begin{bmatrix} 11.671 & 1.003 & 5.5\times10^{-6} \\ 1.003 & 11.671 & 1.003 \\ 5.5\times10^{-6} & 1.003 & 11.671 \end{bmatrix} \begin{bmatrix} a_{-1}' \\ a_0' \\ a_1' \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{-1}' \\ a_0' \\ a_1' \end{bmatrix} = \begin{bmatrix} 0.086 & -0.007 & 0.001 \\ -0.007 & 0.087 & -0.007 \\ 0.001 & -0.007 & 0.086 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}$$

Thus $a_{-1}' = 0.017$ $a_0' = 0.856$ $a_1' = 0.017$

7.40

For the sake of illustration assume $d_1(t)$ is the 7-chip PN sequence shown in Figs 7.38 and 7.39. Then



Note that $d_2(t)$, $d_3(t)$ ... could be shifted versions of this sequence - this permits other transmitters to occupy the same spectral region. Thus, a logical "zero" for transmitter #1 is sent as

(7.40) (Cont.)

$$s_0(t) = A d_i(t) \cos \omega_0 t \qquad t_0 \le t \le t_0 + T$$

and

$$s_1(t) = -A d_i(t) \cos \omega_0 t \qquad t_0 \le t \le t_0 + T$$

(a) Let the bit stream be represented as $m(t)$. Then

$$s_K(t) = A\, m(t)\, d_i(t) \cos(\omega_0 t + \theta) \quad , \text{all } t$$

where

$$m(t) = \sum_{k=-\infty}^{\infty} a_k \Pi\left(\frac{t - kT}{T}\right)$$

and $a_k = \pm 1$ "coin toss" sequence and θ is a random starting phase in $[0, 2\pi]$. The autocorrelation function of $s_K(t)$ is then

$$R_{s_K}(\tau) \triangleq E\{s_K(t)\, s_K(t+\tau)\} = \frac{A^2}{2} R_m(\tau) R_{d_i}(\tau) \cos \omega_0 \tau$$

Assuming $R_m(\tau) = \Lambda(\tau/T)$, the PSD of $m(t)$ is

$$S_m(f) = T \operatorname{Sinc}^2 fT$$

Taking the Fourier transform of $R_{s_K}(\tau)$ to obtain the PSD of $s_K(t)$ gives

$$S_{s_K}(f) = \tfrac{1}{2} A^2 [T \operatorname{Sinc}^2 fT] * \mathcal{F}\{R_{d_i}(\tau) \cos \omega_0 \tau\}.$$

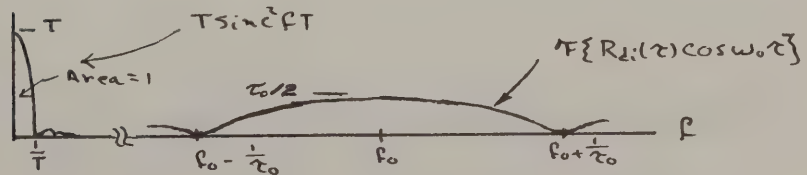
Now assume $d_i(t)$ is an N-chip maximal-length PN sequence with autocorrelation as discussed in text. Then for all practical purposes we can write

$$R_{d_i}(\tau) \approx \Lambda(\tau/\tau_0) \Leftrightarrow \tau_0 \operatorname{Sinc} f\tau_0.$$

Thus

$$\mathcal{F}\{R_{d_i}(\tau) \cos \omega_0 \tau\} = \frac{\tau_0}{2}[\operatorname{Sinc}(f+f_0)\tau + \operatorname{Sinc}(f-f_0)\tau].$$

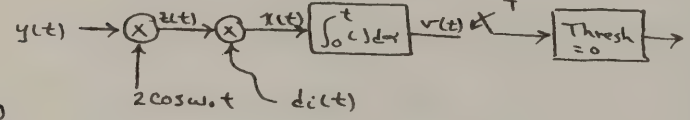
For $T \gg \tau_0 = T/N$, the spectra are as shown.



(7.40) (Cont.)

The convolution of these two spectra reproduces the wide one, i.e., $\mathcal{F}\{R_{xx}(\tau)\cos\omega_0\tau\}$ but scales so that the max. value is now $A^2T\tau_0/4$.

(b)



(c)

$y(t) = A\, m_x(t)\, d_i(t)\cos\omega_0 t + n(t)$

$z(t) = A\, m_x(t)\, d_i(t) + n_c(t) +$ dbl. freqs.

$x(t) = A\, m_x(t) + n_c(t)\, d_i(t) +$ dbl. freqs.

The signal component of $V(T)$ is

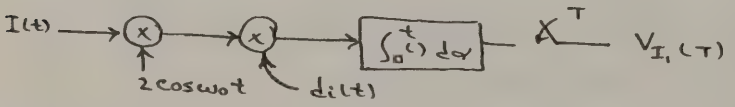
$$V_0 = \int_0^T A\, m_x(t)\, dt = AT$$

and the noise component, N, has variance $\mathrm{Var}(N) = N_0 T$. Thus it follows that

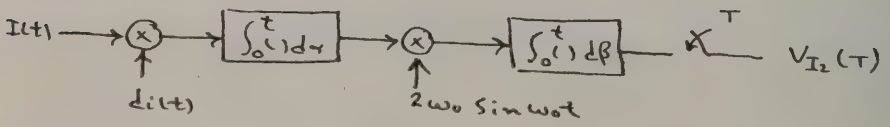
$$P_e = \tfrac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{A^2T}{2N_0}}\right) = \tfrac{1}{2}\,\mathrm{erfc}\sqrt{E/N_0}.$$

The two receivers shown below provide identical performance. However, receiver #2 can provide an improved carrier reference with interference if the input to tracking loop is picked off just after the LPF because interference at that point is wideband.

Receiver #1:



Receiver #2:



(7.40) (Cont.)

Now consider an interference term $I(t) = A_I \cos[(\omega_0+\Delta\omega)t+\theta]$.

$$V_{I_2}(T) = \pm AT + \underbrace{\int_0^T d_i(t)\, n(t)\, 2\cos[(\omega_0+\Delta\omega)t+\theta]}_{N_q} + \underbrace{\int_0^T d_i(t)\, A_I \cos[(\omega_0+\Delta\omega)t+\theta]}_{N_I}$$

$$E\{N_q^2\} = \int_0^T\int_0^T d_i(t)\, d_i(\sigma)\, E\{n(t)n(\sigma)\}\, 4\cos[(\omega_0+\Delta\omega)t+\theta]\cos[(\omega_0+\Delta\omega)\sigma+\theta]$$

$$= \int_0^T d_i^2(t)\, \frac{N_0}{2}\, 4\cos^2\omega_0 t\, dt = N_0 T.$$

$$E\{N_I^2\} = \int_0^T\int_0^T \underbrace{E\{d_i(t)d_i(\sigma)\}}_{=R_d(t-\sigma)\text{ which is nearly an impulse, area}=T_c.} A_I^2 E\{\cos[(\omega_0+\Delta\omega)t+\theta]\cos[(\omega_0+\Delta\omega)\sigma+\theta]\}\,dt\,d\sigma$$

$$= T_c \int_0^T \tfrac{1}{2}A_I^2\, dt = T_c T \frac{A_I^2}{2}.$$

Therefore,

$$\sigma_T^2 = \text{Total noise variance} = N_0 T + T_c T \frac{A_I^2}{2}$$

Now

$$P_e = P_r\{AT + N_q + N_I < 0\}$$

$$= \tfrac{1}{2}\operatorname{erfc}\sqrt{\frac{A^2T^2}{2\sigma_T^2}}.$$

Inserting the expression for σ_T^2 from above

$$P_e = \tfrac{1}{2}\operatorname{erfc}\left[\frac{A^2/2}{N_0\frac{1}{T}+\frac{T_c}{T}\frac{A_I^2}{2}}\right]^{1/2}$$

$$= \tfrac{1}{2}\operatorname{erfc}\left[\frac{P_S}{P_{N_q}+\frac{T_c}{T}P_{N_I}}\right]^{1/2}$$

P_S = signal power.
P_{N_q} = noise power in $\frac{1}{T}$ BW.
P_{N_I} = interference pwr.

Note that we can think of the term $\frac{T_c}{T}$ as the ratio of the modulated bandwidth to interference bandwidth. Thus for spectrum spreading code we might have the term $\ll 1$ so that the interference degradation is made small.

8.1

(a) $f_Z(z/H_1) = f_N(n)\big|_{z=n} = 10e^{-10z}U(z)$.

Given H_2, $Z = S+N$. Since S and N are independent, we can convolve pdfs to find $f_Z(z/H_2)$.

$$f_Z(z/H_2) = f_S(z) * f_N(z) = \int_{-\infty}^{\infty} f_S(z-\alpha) f_N(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} 2e^{-2(z-\alpha)} U(z-\alpha)\, 10e^{-10\alpha} U(\alpha)\, d\alpha \quad \text{for } z>0$$

$$= 20\int_0^z e^{-2z} e^{-8\alpha}\, d\alpha = -2.5e^{-2z}e^{-8\alpha}\Big|_{\alpha=0}^{z} \quad \text{for } z>0$$

$$= 2.5\left(e^{-2z} - e^{-10z}\right)U(z).$$

(b) $\Lambda(z) = f_Z(z/H_2)/f_Z(z/H_1) = 2.5(e^{-2z}-e^{-10z})/10e^{-10z}$

$$= 0.25\left[e^{8z}-1\right].$$

(c) $\eta = \dfrac{P_0[C_{21}-C_{11}]}{Q_0[C_{12}-C_{22}]} = \dfrac{1/4\,(5-0)}{3/4\,(5-0)} = 1/3$

(d) Likelihood ratio test is: $0.25\left[e^{8z}-1\right] \underset{H_1}{\overset{H_2}{\gtrless}} 1/3$.

Simplifying gives $e^{8z} \underset{H_1}{\overset{H_2}{\lessgtr}} 2.33$.

Taking ln. of both sides of equation gives

$$z \underset{H_1}{\overset{H_2}{\gtrless}} 0.106.$$

(e) $P_D = \int_{0.106}^{\infty}\left[2.5e^{-2z} - 2.5e^{-10z}\right]dz = \left[\frac{2.5}{-2}e^{-2z} + \frac{2.5}{10}e^{-10z}\right]_{0.106}^{\infty} = 0.925$

$P_F = \int_{0.106}^{\infty} 10e^{-10z}\, dz = -e^{-10z}\Big|_{0.106}^{\infty} = 0.346$

$\therefore$ Risk $= 1/4\cdot 5 + 3/4\cdot 5\,(1-0.925) - 1/4\,(5)(1-0.346) = 0.714$

(f) Consider the threshold set at η. Then

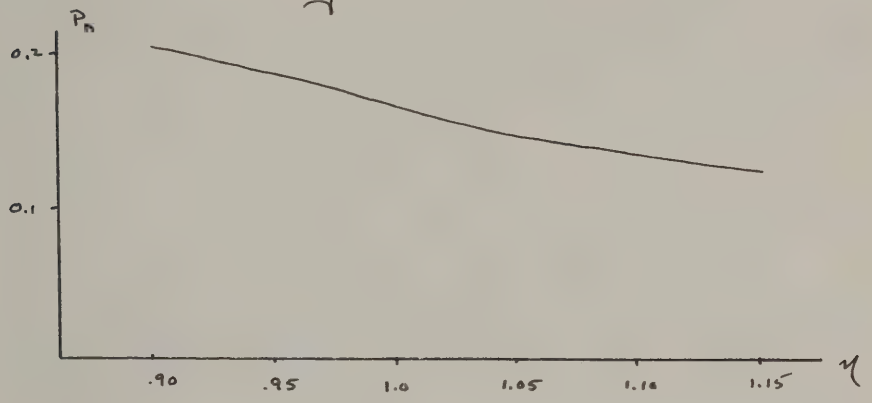
$P_F = e^{-10\eta}$ → Since $P_F \le 10^{-4}$ then $\eta \ge 0.921$.

$P_D = 1.25e^{-2\eta} - 0.25e^{-10\eta}$ (From part (e))

For the values of η of concern (i.e., $\eta \ge 0.921$), $P_D \approx 1.25e^{-2\eta}$. Thus P_D is strictly decreasing with η and so best value of P_D occurs for $\eta = 0.921$ for $P_F \le 10^{-4}$. The resulting P_D is 0.198.

(8.1) (Cont.)

These conclusions are verified by the plot of P_D with increasing η shown below.

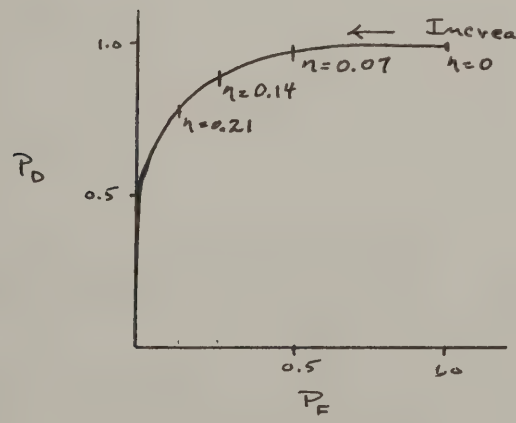


(g) From part (f) we have

$$P_F = e^{-10\eta}, \quad P_D = 1.25e^{-2\eta} - 0.25e^{-10\eta}.$$

From part (d), the likelihood ratio test is

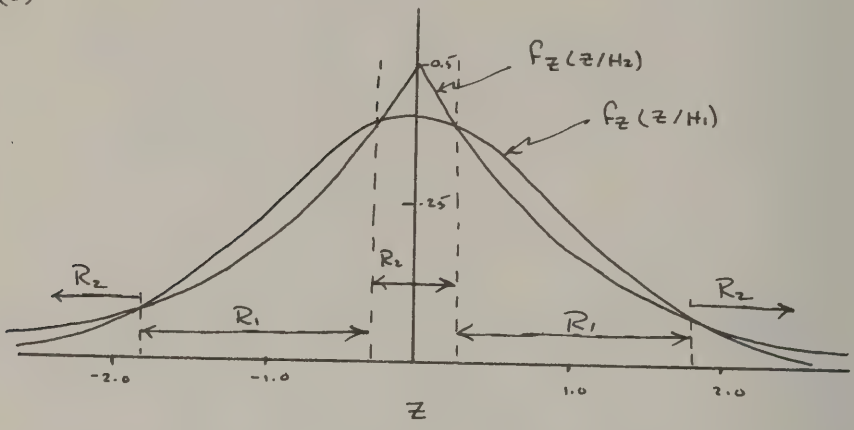
$$0.25\left[e^{8Z} - 1\right] \underset{H_1}{\overset{H_2}{\gtrless}} \eta \quad \Rightarrow \quad Z \underset{H_1}{\overset{H_2}{\gtrless}} \ln(4\eta+1)/8 = \gamma.$$



(8.2)

(a) $$\Lambda(z) = \frac{f_z(z/H_2)}{f_z(z/H_1)} = \frac{\frac{1}{2}e^{-|z|}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}} = \sqrt{\frac{\pi}{2}}\, e^{\frac{1}{2}z^2 - |z|}$$

(b)



Note that the regions R_1, R_2 are for $\eta = 1$.

(8.3)

(a) The region R_2 for each of the thresholds is

$\eta = \eta_1 = 2$: $|z| \geq 2$

$\eta = \eta_2 = 1$: $|z| \geq 2$ or $|z| \leq 1$

$\eta = \eta_3 = 0.5$: ALL z.

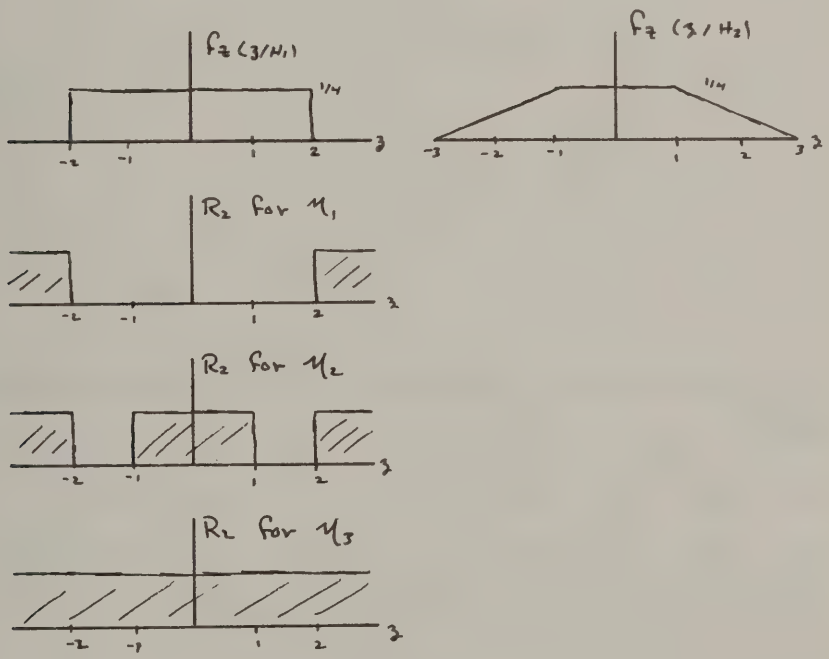
Note that for $\eta_2 = 1$, we have assigned $|z| \leq 1$ to R_2. This is arbitrary and $|z| \leq 1$ could have been assigned to R_1 since $\Lambda(z) = \eta$ here.

The conditional density of the observation is

$f_z(z/H_1) = \frac{1}{4}\Pi(z/4)$, $f_z(z/H_2) = \begin{cases} 0, & |z| > 3 \\ \frac{1}{8}(3-|z|), & 1 < |z| \leq 3 \\ \frac{1}{4}, & |z| \leq 1 \end{cases}$

(8.3) (Cont.)

The probability of false alarm is computed by $P_F = \int_{R_2} f_z(z/H_1)\,dz$. The probability of detection is computed by $P_D = \int_{R_2} f_z(z/H_2)\,dz$. These are easily computed graphically as shown below.

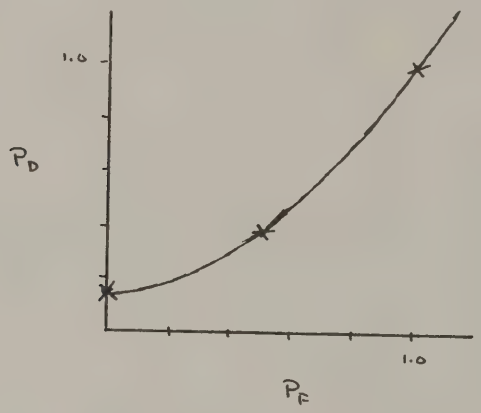


Thus, the following table of values can be computed.

$\mathcal{U}$	P_F	P_D
2	0	.125
1	0.5	.375
0.25	1	1.0

(8.3) (Cont.)

(b) The values of P_D, P_F for the various μ of part (a) can be used to obtain a rough ROC. This is shown below.



(8.4)

Represent $\bar{A}$ by the 3-tuple (a_1, a_2, a_3) and $\bar{B}$ by (b_1, b_2, b_3) where $\{\hat{i}, \hat{j}, \hat{k}\}$ is the standard basis for R^3. Then $\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Define scalar multiplication by $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$ and vector addition by $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3)$.

1) $\alpha_1\bar{A} + \alpha_2\bar{B} = (\alpha_1 a_1 + \alpha_2 b_1, \alpha_1 a_2 + \alpha_2 b_2, \alpha_1 a_3 + \alpha_2 b_3) \in R^3$.

2) $\alpha(\bar{A}+\bar{B}) = \alpha[(a_1+b_1, a_2+b_2, a_3+b_3)]$

$= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3)$

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\alpha b_1, \alpha b_2, \alpha b_3) = \alpha\bar{A} + \alpha\bar{B}$

(8.4) (Cont.)

3) $\alpha_1(\alpha_2\bar{A}) = \alpha_1(\alpha_2 a_1, \alpha_2 a_2, \alpha_2 a_3) = (\alpha_1\alpha_2 a_1, \alpha_1\alpha_2 a_2, \alpha_1\alpha_2 a_3)$

$= (\alpha_1\alpha_2)(a_1, a_2, a_3) = (\alpha_1\alpha_2)\bar{A}$.

4) $1\cdot\bar{A} = (1\cdot a_1, 1\cdot a_2, 1\cdot a_3) = (a_1, a_2, a_3) = \bar{A}$.

5) The unique zero element is $\bar{0} = (0,0,0)$.

$\bar{A} + \bar{0} = (a_1+0, a_2+0, a_3+0) = (a_1, a_2, a_3) = \bar{A}$.

6) $-\bar{A} \triangleq (-a_1, -a_2, -a_3)$. Then $\bar{A} + [-\bar{A}] = 0$.

(8.5)

Consider first the scalar product for energy signals.

$$(x,y) \triangleq \lim_{T\to\infty} \int_{-T}^{T} x(t)\, y^*(t)\, dt .$$

1) $(y,x) = \lim_{T\to\infty} \int_{-T}^{T} y(t)\, x^*(t)\, dt$.

$\therefore (y,x)^* = \lim_{T\to\infty} \int_{-T}^{T} y^*(t)\, x(t)\, dt = (x,y)$.

2) $(\alpha x, y) = \lim_{T\to\infty} \int_{-T}^{T} \alpha\, x(t)\, y^*(t)\, dt = \alpha \lim_{T\to\infty} \int_{-T}^{T} x(t) y^*(t) dt$

$= \alpha(x,y)$.

3) $(x+y, z) = \lim_{T\to\infty} \int_{-T}^{T} [x(t)+y(t)]\, z^*(t)\, dt$

$= \lim_{T\to\infty} \int_{-T}^{T} x(t) z^*(t)\, dt + \lim_{T\to\infty} \int_{-T}^{T} y(t) z^*(t)\, dt$

$= (x,z) + (y,z)$.

4) $(x,x) = \lim_{T\to\infty} \int_{-T}^{T} x(t)\, x^*(t)\, dt = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt$.

The integrand is ≥ 0 for all t. If $x(t)$ is continuous, if $x(t) \neq 0$ for $t = t_1$, $x(t) \neq 0$ for some ϵ-neighborhood and so integral > 0.

The scalar product for power signals is considered in the same manner as above only $\frac{1}{2T}$ is placed in front of integral.

8.6

(a) $(x_1, x_2) = \lim_{T\to\infty} \int_0^T 2e^{-ct}\,dt = 1/3.$

(b) $(x_1, x_2) = \lim_{T\to\infty} \int_0^T e^{-st} e^{-sjt}\,dt = \frac{1}{s+js}.$

(c) $(x_1, x_2) = \lim_{T\to\infty} \frac{1}{2T} \int_0^T (\cos 2\pi t)\,\frac{1}{2}(1-\cos 4\pi t)\,dt = 0.$

(d) $(x_1, x_2) = \lim_{T\to\infty} \frac{1}{2T} \int_0^T 10\cos 2\pi t\,dt = 0.$

8.7

$$\|x_1(t) + x_2(t)\|^2 = \lim_{T\to\infty} \int_{-T}^{T} (x_1(t) + x_2(t))^2\,dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} x_1^2(t)\,dt + 2\lim_{T\to\infty} \int_{-T}^{T} x_1(t)x_2(t)\,dt + \lim_{T\to\infty} \int_{-T}^{T} x_2^2(t)\,dt$$

$$= \|x_1(t)\|^2 + 2(x_1(t), x_2(t)) + \|x_2(t)\|^2.$$

We see that if $x_1(t)$, $x_2(t)$ are orthogonal, then the norm squared of the sum equals the sum of norms squared. Conversely, if the norm squared of sum equals the sum of norms squared, then signals are orthogonal.

8.8

$$\|x_1\|^2 = \int_{-\infty}^{\infty} x_1^2(t)\,dt = \int_{-1}^{1} dt = 2 \qquad \|x_1\| = \sqrt{2}$$

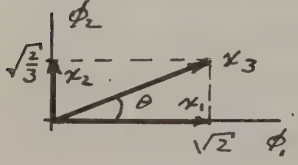
$$\|x_2\|^2 = \int_{-\infty}^{\infty} x_2^2(t)\,dt = \int_{-1}^{1} t^2\,dt = 2/3, \qquad \|x_2\| = \sqrt{2/3}$$

$$\|x_3\| = \int_{-\infty}^{\infty} x_3^2(t)\,dt = \int_{-1}^{1} (1+t)^2\,dt = 8/3, \qquad \|x_3\| = \sqrt{8/3}$$

$$(x_1, x_2) = \int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \int_{-1}^{1} t\,dt = 0$$

$$(x_3, x_1) = \int_{-\infty}^{\infty} x_3(t)x_1(t)\,dt = \int_{-1}^{1} (1+t)\,dt = 2$$

Since $(x_1, x_2) = 0$, x_1 and x_2 are orthogonal. Thus choose x_1 and x_2 as basis functions.



$$\|x_3\|^2 = 2 + \frac{2}{3} = 8/3, \quad \|x_3\| = \sqrt{\frac{8}{3}}$$

Also:

$$\cos\theta = \frac{\|x_1\|}{\|x_3\|} = \frac{\sqrt{2}}{\sqrt{8/3}} = \sqrt{\frac{3}{4}}$$

Using (x_3, x_1):

$$\cos\theta = \frac{(x_3, x_1)}{\|x_1\|\,\|x_3\|} = \sqrt{\frac{3}{4}}$$

(8.9)

We want to show that $|(\chi_1, \chi_2)|^2 \le \|\chi_1\|^2 \cdot \|\chi_2\|^2$ where $\chi_1(t)$, $\chi_2(t)$ are expressed in terms of an orthonormal basis.

Let $\chi_1(t) = \sum_{n=1}^{N} a_n \phi_n(t)$, $\chi_2(t) = \sum_{n=1}^{N} b_n \phi_n(t)$.

Then

$$(\chi_1, \chi_2) = \left(\sum_{n=1}^{N} a_n \phi_n, \sum_{m=1}^{N} b_m \phi_m \right) = \sum_{n=1}^{N} b_n^* a_n (\phi_n, \phi_n)$$
$$= \sum_{n=1}^{N} a_n b_n^*$$

where the last two steps follow from the fact that $(\phi_n, \phi_m) = \delta_{nm}$.

In a similar fashion it follows that

$$\|\chi_1\|^2 = \sum_{n=1}^{N} a_n a_n^* = \sum_{n=1}^{N} |a_n|^2 ; \quad \|\chi_2\|^2 = \sum_{n=1}^{N} b_n b_n^* = \sum_{n=1}^{N} |b_n|^2.$$

Thus we must show that

$$\left| \sum_{n=1}^{N} a_n b_n^* \right|^2 \le \left[\sum_{n=1}^{N} a_n a_n^* \right] \left[\sum_{n=1}^{N} b_n b_n^* \right].$$

The left side of inequality is simplified first:

$$\left| \sum_{n=1}^{N} a_n b_n^* \right| = \left[\sum_{j=1}^{N} a_j b_j^* \right] \left[\sum_{K=1}^{N} a_K b_K^* \right]^*$$
$$= \left[\sum_{j=1}^{N} a_j b_j^* \right] \left[\sum_{K=1}^{N} a_K^* b_K \right] = \sum_{j=1}^{N} \sum_{K=1}^{N} a_j a_K^* b_j^* b_K.$$

The right side of the inequality is written

$$\left[\sum_{n=1}^{N} a_n a_n^* \right] \left[\sum_{m=1}^{N} b_m b_m^* \right] = \sum_{j=1}^{N} \sum_{K=1}^{N} a_j a_j^* b_K^* b_K .$$

Combining the previous results gives the following inequality which we need to show true.

$$\sum_{j=1}^{N} \sum_{K=1}^{N} a_j a_j^* b_K b_K^* \ge \sum_{j=1}^{N} \sum_{K=1}^{N} a_j a_K^* b_j^* b_K .$$

This is equivalent to

$$\sum_{j=1}^{N} \sum_{K=1}^{N} \left[2 a_j a_j^* b_K b_K^* - 2 a_j a_K^* b_j^* b_K \right] \ge 0$$

or

$$\sum_{j=1}^{N} \sum_{K=1}^{N} \left[a_j a_j^* b_K b_K^* - 2 a_j a_K^* b_j^* b_K + a_K a_K^* b_j b_j^* \right] \ge 0.$$

8.9 (cont.)

This last expression follows since

$$\sum_{j=1}^{N}\sum_{k=1}^{N} a_j a_j^* b_k b_k^* = \sum_{j=1}^{N}\sum_{k=1}^{N} a_k a_k^* b_j b_j^*$$

The expression can be factored to give

$$\sum_{j=1}^{N}\sum_{k=1}^{N} |a_j b_k - a_k b_j|^2 \geq 0 .$$

But this last inequality is obviously true since each term in sum is ≥ 0. This verifies Schwartz's inequality when waveforms are expressed in terms of a finite dimensional basis.

8.10

(a) We obtain $\phi_1(t)$ by normalizing $V_1(t) = S_1(t)$.

$\|S_1\| = \sqrt{2} \Rightarrow \phi_1(t) = \frac{1}{\sqrt{2}} S_1(t)$.

$(S_2, \phi_1) = \frac{1}{\sqrt{2}}$ so $V_2(t) = S_2(t) - \frac{1}{\sqrt{2}}\phi_1(t)$

$= S_2(t) - \frac{1}{2} S_1(t)$

is orthogonal to $\phi_1(t)$. We obtain $\phi_2(t)$ by normalizing $V_2(t)$. $\|V_2\| = \sqrt{\frac{3}{2}}$ so $\phi_2(t) = \sqrt{\frac{2}{3}}\left(S_2(t) - \frac{1}{2}S_1(t)\right)$.

$(S_3, \phi_1) = \sqrt{2}$, $(S_3, \phi_2) = \sqrt{2/3}$ so

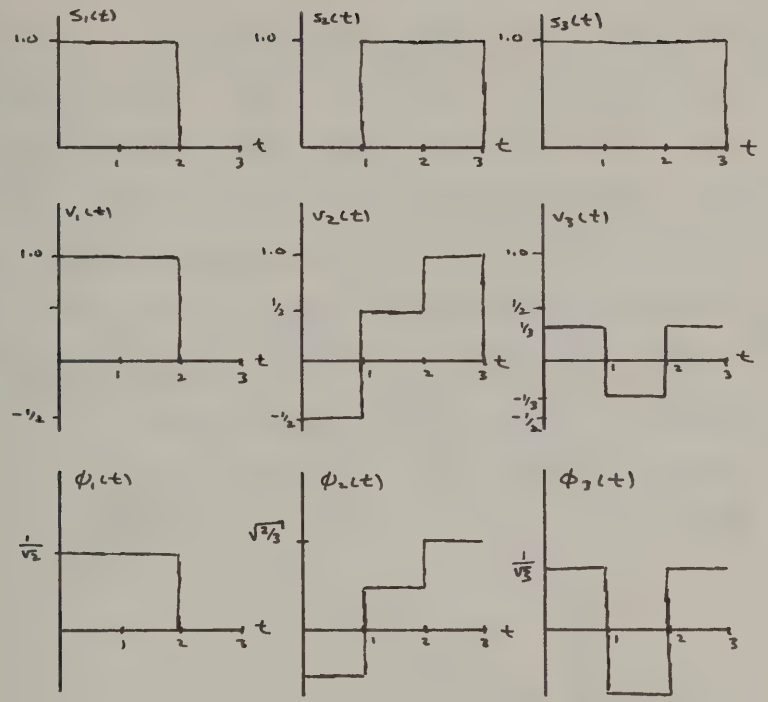
$V_3(t) = S_3(t) - \sqrt{2}\,\phi_1(t) - \sqrt{\frac{2}{3}}\,\phi_2(t) = S_3(t) - \frac{2}{3}S_1(t) - \frac{2}{3}S_2(t)$

is orthogonal to both $\phi_1(t)$ and $\phi_2(t)$. We obtain $\phi_3(t)$ by normalizing $V_3(t)$.

$\|V_3(t)\| = \frac{1}{\sqrt{3}}$ so $\phi_3(t) = \sqrt{3}\left[S_3(t) - \frac{2}{3}S_1(t) - \frac{2}{3}S_2(t)\right]$.

The various signals involved in the Gram-Schmidt procedure just performed are shown below.

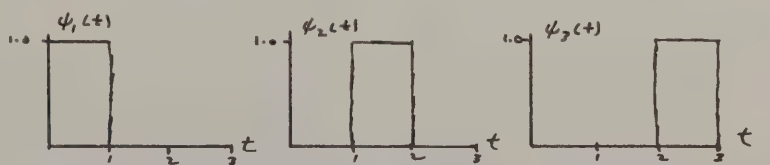
8.10 (Cont.)



(b) s_1, s_2, s_3 are expressed in terms of the basis $\{\phi_1, \phi_2, \phi_3\}$ by evaluating $\{s_i, \phi_j\}$. The results are

$$s_1 = \sqrt{2}\,\phi_1 \quad , \quad s_2 = \frac{1}{\sqrt{2}}\phi_1 + \sqrt{3/2}\,\phi_2 \quad , \quad s_3 = \sqrt{2}\phi_1 + \sqrt{2/3}(\phi_2 + \phi_3).$$

Note that a basis could have been obtained by inspection. This is done by noting that the functions ψ_1, ψ_2, ψ_3, as defined below are orthonormal and span the set $\{s_1, s_2, s_3\}$.



8.11

(a) $s_1(t) = \sqrt{2}A\cos(2\pi f_0 t + \pi/2)$ for $0 \le t \le N/f_0$

$s_2(t) = \sqrt{2}A\cos(2\pi f_0 t + \pi)$ for $0 \le t \le N/f_0$

$s_3(t) = \sqrt{2}A\cos(2\pi f_0 t + 3\pi/2)$ for $0 \le t \le N/f_0$

$s_4(t) = \sqrt{2}A\cos(2\pi f_0 t + 2\pi)$ for $0 \le t \le N/f_0$.

Note that $s_3(t) = -s_1(t)$, $s_4(t) = -s_2(t)$ and $s_1(t)$, $s_2(t)$ are orthogonal.

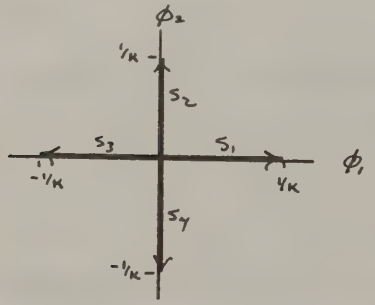
$$\int_0^{N/f_0} s_1^2(t)\,dt = \int_0^{N/f_0} s_2^2(t)\,dt = 2A^2\int_0^{N/f_0}\cos^2(2\pi f_0 t)\,dt = A^2 N/f_0.$$

Thus, the orthonormal basis is

$\phi_1(t) = K\,s_1(t)$

$\phi_2(t) = K\,s_2(t)$ where $K = \sqrt{\frac{f_0}{NA^2}}$.

(b)



(8.12)

$$s_i(t) = A\cos(\omega_0 t + i\,2\pi/M) \qquad i = 1,2,\ldots,M \;;\; 0 \le t \le T.$$

(a) A suitable basis is

$$\phi_1(t) = \sqrt{\tfrac{2}{T}}\cos\omega_0 t \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\tfrac{2}{T}}\sin\omega_0 t \qquad 0 \le t \le T$$

This can be verified by noting that $\phi_1(t)$, $\phi_2(t)$ are orthogonal and are a minimum spanning set. The signal space dimension is two.

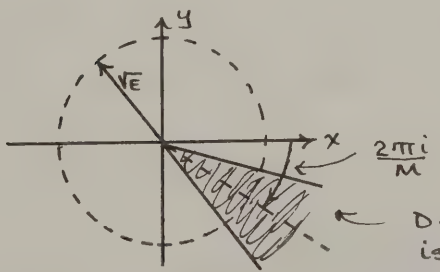
$$x_i = \int_0^T s_i(t)\,\phi_1(t)\,dt = A\sqrt{\tfrac{T}{2}}\cos\frac{2\pi i}{M} = \sqrt{E}\cos\frac{2\pi i}{M}$$

$$y_i = \int_0^T s_i(t)\,\phi_2(t)\,dt = -A\sqrt{\tfrac{T}{2}}\sin\frac{2\pi i}{M} = -\sqrt{E}\sin\frac{2\pi i}{M}$$

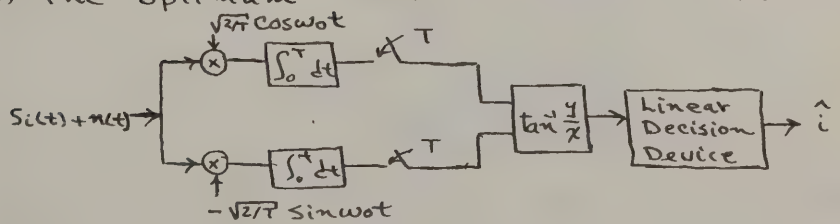
and so

$$s_i(t) = x_i\,\phi_1(t) + y_i\,\phi_2(t)$$
$$= \sqrt{E}\cos\frac{2\pi i}{M}\,\phi_1(t) - \sqrt{E}\sin\frac{2\pi i}{M}\,\phi_2(t).$$

(b) The optimum partitioning of the signal space is a wedge as shown below.



(c) The optimum receiver is shown below.



(8.12) (Cont.)

(d) The probability of error conditioned on $s_i(t)$ transmitted is found by computing the probability of correct decision which is found by forming the conditional joint pdf of (X, Y) and integrating over the decision region for $s_i(t)$, R_i. Then the probability of error is found using

$$P(\text{Error}/s_i(t)) = 1 - P(\text{Correct}/s_i(t))$$

Thus

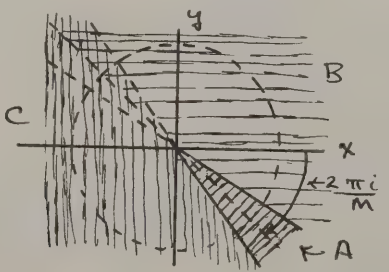
$$P(\text{Error}/s_i(t)) = 1 - \frac{1}{2\pi N_0} \iint_{R_i} e^{-\frac{1}{N_0}\left[(x-\sqrt{E}\cos\frac{2\pi i}{M})^2 + (y+\sqrt{E}\sin\frac{2\pi i}{M})^2\right]} dx\,dy$$

Using change of variables with $x = r\sqrt{N_0}\cos\theta$ $y = r\sqrt{N_0}\sin\theta$ gives

$$P(\text{Error}/s_i(t)) = 1 - \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \int_0^{\infty} r\, e^{-\frac{1}{2}\left[r^2 - 2r\sqrt{\frac{2E}{N_0}}\cos\theta + \frac{2E}{N_0}\right]} dr\,d\theta$$

which is independent of i and so is also the average probability of error.

(e) Suppose we transmit $s_i(t)$. Then the signal space is shown below.



Note that if the received signal (X,Y) falls in region A we detect properly, and the

(8.12) (Cont.)

probability of error is $P\{(X,Y) \notin A\}$. Clearly,

$$P(E) > P[(X,Y) \in B]$$

$$P(E) < P[(X,Y) \in B] + P[(X,Y) \in C].$$

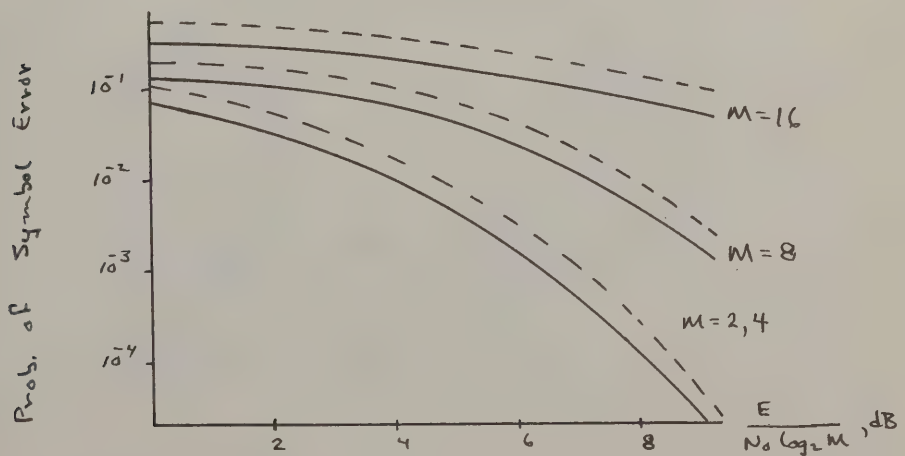
But these probabilities are particularly easy to compute since the received signal (X,Y) will fall in B or C only if the noise component perpendicular to the boundary of the half planes exceeds $\sqrt{E}\sin\pi/M$. This is simply

$$P\{(X,Y) \in B\} = P\{(X,Y) \in C\} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E}{N_0}}\sin\pi/m}^{\infty} e^{-x^2/2}\,dx$$

$$= \frac{1}{2} \int_{\sqrt{E/N_0}\sin\pi/m}^{\infty} \frac{2}{\sqrt{\pi}} e^{-y^2}\,dy$$

$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{E/N_0}\sin\pi/M\right].$$

Define $\gamma = SNR = E/N_0$. Then the bound is $\frac{1}{2}\operatorname{erfc}[\sqrt{\gamma}\sin\pi/M] \le P(E) \le \operatorname{erfc}[\sqrt{\gamma}\sin\pi/M]$. Note lower bound exact for $M=2$. The plot for various M follows.



(8.12) (Cont.)

Note that the figure shows the P_e plotted versus $E/N_0 \log_2 M$. This is so that results are normalized to give a constant information rate. The curves show that PSK does not provide improved performance in noise with increased M. (This assumes that symbol error for any M causes same difficulty.) However, we may be operating in a bandwidth constrained system and so $M > 4$ may be useful.

(8.13)

$$P_e = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \int_{-\infty}^{y+\sqrt{E/N_0}} \frac{1}{\sqrt{\pi}} e^{-x^2} dx\, dy.$$

Using change of variables with $z = x - y$ gives

$$P_e = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \int_{-\infty}^{\sqrt{E/N_0}} \frac{1}{\sqrt{\pi}} e^{-(z+y)^2} dz\, dy.$$

Changing the order of integration gives

$$P_e = \int_{-\infty}^{\sqrt{E/N_0}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} e^{-(z+y)^2} dy\, dz$$

$$= \int_{-\infty}^{\sqrt{E/N_0}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-2y^2 - z^2 - 2yz}\, dy\, dz.$$

Completing the square on y, and simplifying gives

$$P_e = \int_{-\infty}^{\sqrt{E/N_0}} \frac{1}{\sqrt{\pi}} e^{-z^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-2(y+z/2)^2} dy\, dz.$$

It can be shown that $\int_{-\infty}^{\infty} e^{-a^2(t-v)^2} dt = \sqrt{\pi}/a$.

Thus,

$$P_e = \int_{-\infty}^{\sqrt{E/N_0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} + \frac{1}{2} \int_{0}^{\sqrt{E/2N_0}} \frac{2}{\sqrt{\pi}} e^{-z^2} dz$$

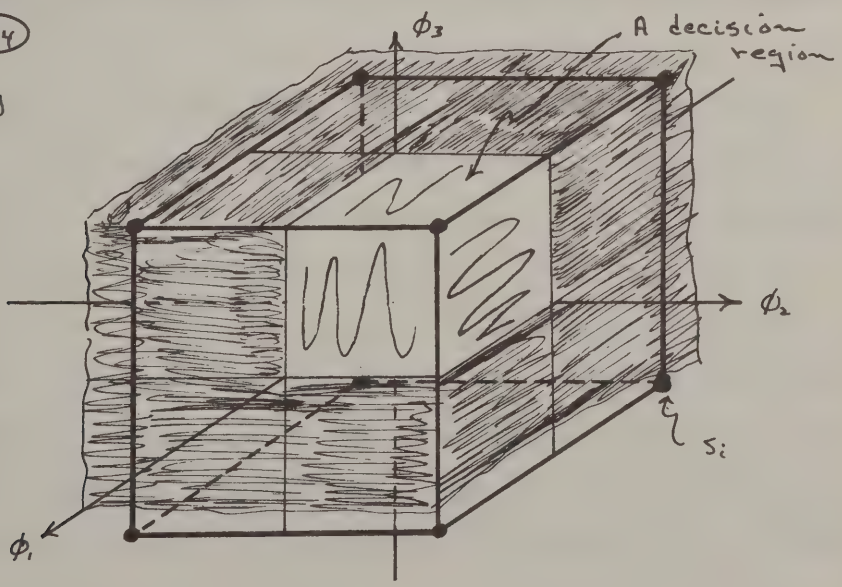
$$= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{E}{2N_0}}\right) = 1 - \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{E}{2N_0}}\right).$$

$$\therefore \quad P_e = 1 - P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right).$$

This is the same as binary, coherent FSK.

(8.14)

(a)



(8.14) (Cont.)

Note that the optimum partitions result in 1/8-th of 3-space decision regions.

(b) A possible choice of α_{ik} gives

$\bar{s}_0 = (-1, -1, -1)$ $\quad$ $\bar{s}_4 = (1, -1, -1)$

$\bar{s}_1 = (-1, -1, 1)$ $\quad$ $\bar{s}_5 = (1, -1, 1)$

$\bar{s}_2 = (-1, 1, -1)$ $\quad$ $\bar{s}_6 = (1, 1, -1)$

$\bar{s}_3 = (-1, 1, 1)$ $\quad$ $\bar{s}_7 = (1, 1, 1)$

If $\bar{s}_i$ is sent, the received vector $\bar{y}$ is

$$\bar{y} = \bar{s}_i + \bar{N}$$

where $\bar{N} = (n_1, n_2, n_3)$ and the n_i are independent Gaussian random variables with variance $N_0/2$. We use maximum likelihood detection (which is equivalent to minimum error probability for equally likely signals) and so the decision rule is:

Choose $\bar{s}_i$ as transmitted if $P(\bar{y}/\bar{s}_i) \geq P(\bar{y}/\bar{s}_j)$ $\forall j$.

Since the conditional pdf of $\bar{y}$ is

$$f_{\bar{y}}(\bar{y}/s_i) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^3 \exp\left[-\frac{1}{N_0}\|\bar{y} - \bar{s}_i\|^2\right],$$

the decision rule reduces to minimum distance detection and is:

Choose $\bar{s}_i$ as transmitted if

$\|\bar{y} - \bar{s}_i\|^2 \leq \|\bar{y} - \bar{s}_j\|^2$ $\forall j$.

From the figure of the signal vectors we see that the problem has symmetry with respect to distances between signals. Thus, the conditional probabilities of error

(8.14) (Cont.)

are all the same. This enables us to find the average probability of error from the conditional probability of error since they are equal.

Let's compute $P(\text{Error}/\bar{s}_0)$. Note that the most likely error is to choose $\bar{s}_1$, $\bar{s}_2$, or $\bar{s}_4$ since they are closest. Also, we make a correct decision only if

$$\|\bar{y}-\bar{s}_0\|^2 < \|\bar{y}-\bar{s}_1\|^2$$
$$\|\bar{y}-\bar{s}_0\|^2 < \|\bar{y}-\bar{s}_2\|^2$$
$$\|\bar{y}-\bar{s}_0\|^2 < \|\bar{y}-\bar{s}_4\|^2.$$

This is a necessary and sufficient condition for correct decision. Thus

$$(y_1+1)^2+(y_2+1)^2+(y_3+1)^2 \le (y_1+1)^2+(y_2+1)^2+(y_3-1)^2$$
$$" \quad " \quad " \le (y_1+1)^2+(y_2-1)^2+(y_3+1)^2$$
$$" \quad " \quad " \le (y_1-1)^2+(y_2+1)^2+(y_3+1)^2$$

which reduces to

$$(y_3+1)^2 \le (y_3-1)^2,\ (y_2+1)^2 \le (y_2-1)^2,\ (y_1+1)^2 \le (y_1-1)^2$$

which is equivalent to

$$y_1 \le 0,\quad y_2 \le 0,\quad y_3 \le 0.$$

Therefore, $P(\text{Correct}/\bar{s}_0) = P_r(y_1 \le 0, y_2 \le 0, y_3 \le 0)$. The derivation of this probability is simplified because the y_i are independent and identically distributed. The derivation follows.

(8.14) (Cont.)

$$P_r(\text{Correct}/\bar{s}_0) = [P(y_1 \le 0)]^3 = \left[\frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{0} e^{-\frac{1}{N_0}(y_1-\sqrt{E/3})^2} dy_1\right]^3$$

$$= \left[\frac{1}{2}\int_{-\infty}^{\sqrt{E/3N_0}} \frac{2}{\sqrt{\pi}} e^{-y_1^2} dy_1\right]^3 = \left[\tfrac{1}{2} + \tfrac{1}{2}\,\text{erf}\left(\sqrt{\tfrac{E}{3N_0}}\right)\right]^3$$

$$= \left[1 - \tfrac{1}{2}\,\text{erfc}\left(\sqrt{\tfrac{E}{3N_0}}\right)\right]^3.$$

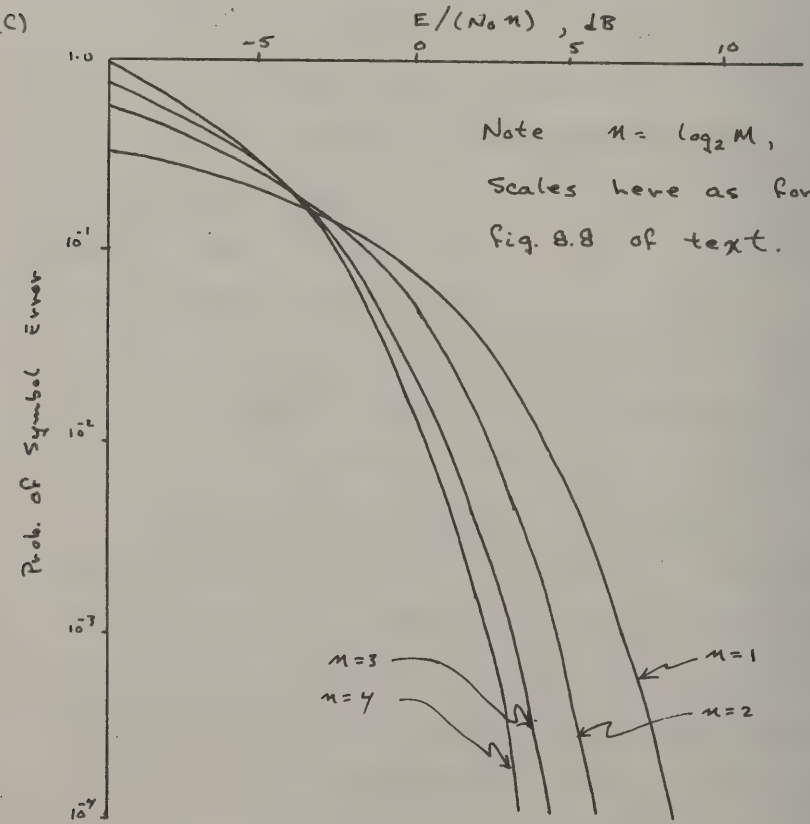
$$\therefore P_r(\text{Error}/\bar{s}_0) = P_r(\text{Error}) = 1 - \left[1 - \tfrac{1}{2}\,\text{erfc}\left(\sqrt{\tfrac{E}{3N_0}}\right)\right]^3.$$

The generalization to n-dimensions follows immediately since a necessary and sufficient condition for correct reception of $\bar{s}_0$ is

$$y_1 \le 0,\ y_2 \le 0,\ \ldots,\ y_n \le 0.$$

$$\therefore P_E = 1 - \left[1 - \tfrac{1}{2}\,\text{erfc}\left(\sqrt{\tfrac{E}{nN_0}}\right)\right]^n$$

(c)



(8.14) (Cont.)

Comparison of the figure above for hypercube signalling with fig. 8.8 which is for orthogonal signalling reveals that hypercube signalling is about 3 db inferior for $n=1,2$ but more closely matches performance of orthogonal signalling for larger n.

(8.15)

(a) $s_i(t) = A\cos(\omega_0 + i\Delta\omega)t \quad , \quad 0 \le t \le T.$

$$\int_0^T s_i(t)\, s_j(t)\, dt = \int_0^T A^2 \cos(\omega_0 + i\Delta\omega)t \cdot \cos(\omega_0 + j\Delta\omega)t\, dt$$

$$= \frac{A^2}{2}\int_0^T \left[\cos(2\omega_0 + (i+j)\Delta\omega)t + \cos(i-j)\Delta\omega t\right] dt$$

$$= \frac{A^2}{2}\left[\frac{1}{2\omega_0 + (i+j)\Delta\omega}\sin[2\omega_0 + (i+j)\Delta\omega]t + \frac{1}{(i-j)\Delta\omega}\sin(i-j)\Delta\omega t\right]\Bigg|_0^T$$

$$= \frac{A^2}{2}\left[\frac{1}{2\omega_0 + (i+j)\Delta\omega}\sin[2\omega_0 + (i+j)\Delta\omega] + \frac{1}{(i-j)\Delta\omega}\sin(i-j)\Delta\omega T\right]$$

where it is assumed that $i \neq j$. Note that first term in the last expression can be neglected because the $2\omega_0$ in denominator makes it very small. Thus

$$\int_0^T s_i(t)\, s_j(t)\, dt = \frac{A^2}{2}\frac{1}{(i-j)\Delta\omega}\sin(i-j)\Delta\omega T.$$

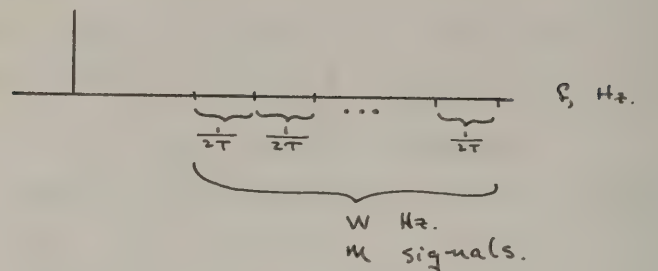
We want this term zero so $\sin(i-j)\Delta\omega T = 0$.

So $(i-j)\Delta\omega T = n\pi$, and since $\Delta\omega = 2\pi\Delta f$,

$(i-j)\,2\pi\,\Delta f\, T = n\pi \Rightarrow \Delta f = \frac{n}{(i-j)2T}$. The minimum n is 1, and minimum $i-j$ is 1, so, $\Delta f_{min} = \frac{1}{2T}$.

(8.15) (Cont.)

(b)



$\frac{1}{2T}$ Hz/signal, W Hz bandwidth, so $M = W/\frac{1}{2T}$ or $M = 2WT$.

(c) Use $\phi_i(t) = \sqrt{\frac{2}{T}} \cos(\omega_0 + i\Delta\omega)t$, $0 \le t \le T$, for orthogonal functions in vertices-of-hypercube signalling. Then in W Hz., we can have $N = 2WT$ $\phi_i(t)$'s. But $M = 2^N$ (see Prob. 8.14) so $M = 2^{2WT}$.

(8.16)

$$P(E/H_1) = \int_0^\infty \left[\int_{r_1}^\infty f_{R_2}(r_2/H_1)\, dr_2 \right] f_{R_1}(r_1/H_1)\, dr_1$$

where

$$f_{R_1}(r_1/H_1) = \frac{2r_1}{2E\sigma^2 + N_0} e^{-r_1^2/(2E\sigma^2 + N_0)}$$

and

$$f_{R_2}(r_2/H_1) = \frac{2r_2}{N_0} e^{-r_2^2/N_0}.$$

First consider the term in brackets in the expression for $P(E/H_1)$.

$$\int_{r_1}^\infty f_{R_2}(r_2/H_1)\, dr_2 = \int_{r_1}^\infty \frac{2r_2}{N_0} e^{-r_2^2/N_0}\, dr_2 = \int_{r_1^2/N_0}^\infty e^{-\xi}\, d\xi$$

where change of variables with $\xi = r_2^2/N_0$ was used. Therefore,

$$\int_{r_1}^\infty f_{R_2}(r_2/H_1)\, dr_2 = e^{-r_1^2/N_0}.$$

(8.16) (Cont.)

Thus,

$$P(E/H_1) = \int_0^\infty e^{-r_1^2/N_0} \cdot \frac{2r_1}{2E\sigma^2+N_0} e^{-r_1^2/(2E\sigma^2+N_0)} dr_1$$

$$= \int_0^\infty \frac{2r_1}{2E\sigma^2+N_0} e^{-r_1^2/K} dr_1 \quad \text{where} \quad K = \frac{N_0(2E\sigma^2+N_0)}{2E\sigma^2+2N_0}$$

$$= \frac{K}{2E\sigma^2+N_0} \underbrace{\int_0^\infty \frac{2r_1}{K} e^{-r_1^2/K} dr_1}_{=1}$$

$$= \frac{K}{2E\sigma^2+N_0} = \frac{1}{2} \cdot \frac{1}{1+E\sigma^2/N_0}.$$

(8.17)

Given the i^{th} hypothesis we receive

$$y(t) = G_i\sqrt{\frac{2E_i}{T}} \cos(\omega_i t + \Theta_i)$$

where $i = 1, 2, \ldots, M$ and $0 \le t \le T$.

G_i is Rayleigh, Θ_i is uniform $[0, 2\pi]$, E_i is the energy of the i^{th} signal of duration T, and the ω_i are chosen so that the signals are orthogonal.

(a) We assume slow fading and expand $y(t)$ into a $2M$ dimensional signal space where the basis is

$$\phi_{ci}(t) = \sqrt{2/T}\cos(\omega_i t) \qquad i = 1, 2, \ldots, M$$

$$\phi_{si}(t) = \sqrt{2/T}\sin(\omega_i t) \qquad 0 \le t \le T$$

Thus, we base our decision on the vector z given by

(8.17) (Cont.)

$$\bar{z} = (z_{c1}, z_{s1}, z_{c2}, z_{s2}, \ldots, z_{cm}, z_{sm})$$

where

$$z_{ci} = (y, \phi_{ci}) = \int_0^T y(t)\,\phi_{ci}(t)\,dt$$

$$z_{si} = (y, \phi_{si}) = \int_0^T y(t)\,\phi_{si}(t)\,dt .$$

Suppose that the i^{th} hypothesis is given. Then the components of $\bar{z}$ are given by

$$z_{cj} = \sqrt{E_j}\,G_{cj} + N_{cj} \qquad i = j$$

$$= N_{cj} \qquad i \neq j$$

$$z_{sj} = \sqrt{E_j}\,G_{sj} + N_{sj} \qquad i = j$$

$$= N_{sj} \qquad i \neq j$$

where $G_{ci} = G_i \cos\theta$, $G_{si} = -G_i \sin\theta$ are independent Gaussian random variables with variance σ^2, and N_{ci}, N_{si} are independent Gaussian random variables with variance $N_0/2$. Note that $\bar{z}$ is jointly Gaussian and the pdf of $\bar{z}$ given hypothesis i is

$$f_{\bar{z}}(\bar{z}/H_i) = \frac{\exp\left(\frac{-z_{ci}^2 + z_{si}^2}{E_i\sigma^2 + N_0}\right)}{\pi^M (E_i\sigma^2 + N_0) N_0^M} \exp\left(-\frac{1}{N_0} \sum_{\substack{j=1 \\ j \neq i}}^{M} z_{cj}^2 + z_{sj}^2\right).$$

The decision rule that minimizes the probability of error is maximum likelihood since we assume hypotheses equally-likely. Thus we decide hypothesis i sent if

(8.17) (Cont.)

$f_{\bar{z}/H_i}(\bar{z}/H_i) \geq f_{\bar{z}/H_j}(\bar{z}/H_j)$ $\forall j$. Alternately, we can examine $\log f_{\bar{z}/H_i}(\bar{z}/H_i)$ and our decision criterion becomes decide hypothesis i if

$$\frac{z_{ci}^2 + z_{si}^2}{E_i\sigma^2 + N_0} + \frac{1}{N_0}\sum_{\substack{K=1 \\ K\neq i}}^{M}(z_{cK}^2 + z_{sK}^2) \leq \frac{z_{cj}^2 + z_{sj}^2}{E_j\sigma^2 + N_0} + \frac{1}{N}\sum_{\substack{K=1 \\ K\neq j}}^{M} z_{cK}^2 + z_{sK}^2 \quad \forall j$$

This expression can be simplified somewhat. Let $S^2 = \sum_{L=1}^{M}(z_{cL}^2 + z_{sL}^2)$. Then

$$\frac{z_{ci}^2 + z_{si}^2}{E_i\sigma^2 + N_0} + \frac{1}{N_0}\sum_{\substack{K=1 \\ K\neq i}}^{M}(z_{cK}^2 + z_{sK}^2) = \frac{S^2}{N_0} - \frac{E_i\sigma^2}{N_0(E_i\sigma^2 + N_0)}(z_{ci}^2 + z_{si}^2)$$

Therefore the decision rule is choose that i corresponding to the maximum of

$$\frac{E_i\sigma^2}{E_i\sigma^2 + N_0}(z_{ci}^2 + z_{si}^2)$$

If the E_i are equal then the optimum receiver is an M signal replica of the two signal receiver in Fig. 8.9. Note that if the energies are not equal then the output of the i^{th} summer should be scaled by

$$\frac{E_i\sigma^2 + N_0}{E_i\sigma^2}$$

(b) Assume the E_i are equal. We note

(8.17) (Cont.)

that $z_i = \sqrt{z_{ci}^2 + z_{si}^2}$ is Rayleigh given any H_i. Without loss of generality, suppose H_1 is given. Then

$$f_{z_1/H_1}(z_1/H_1) = \frac{z_1}{E\sigma^2+N_0} \exp\left[-z_1^2/2(E\sigma^2+N_0)\right]$$

$$f_{z_i/H_1}(z_i/H_1) = \frac{z_i}{N_0} \exp(-z_i^2/2N_0) \quad i \neq 1.$$

It follows that the probability of a correct decision is $\Pr\{z_i \leq z_1 \ \forall i = 1,2,\ldots,M\}$. Thus

$$\Pr\{\text{Correct}/H_1\} = \int_0^\infty \frac{z_1}{E\sigma^2+N_0} e^{-z_1^2/2(E\sigma^2+N_0)} \left[\int_0^{z_1} \frac{z_2}{N_0} e^{-z_2^2/2N_0} dz_2\right]^{M-1} dz_1$$

This expression follows because the z_i are independent. We can write

$$\int_0^{z_1} \frac{z_2}{N_0} e^{-z_2^2/2N_0} dz_2 = 1 - e^{-z_1^2/2N_0}$$

and so the expression becomes

$$\Pr\{\text{Correct}/H_1\} = \int_0^\infty \frac{z_1}{E\sigma^2+N_0} e^{-z_1^2/2(E\sigma^2+N_0)} \left[1 - e^{-z_1^2/2N_0}\right]^{M-1} dz_1$$

The binomial theorem states

$$(x_1 + x_2)^n = \sum_{i=0}^{n} \binom{n}{i} x_1^i x_2^{n-i}$$

So we can write

$$\left[1 - e^{-z_1^2/2N_0}\right]^{M-1} = \sum_{i=0}^{M-1} \binom{M-1}{i} (1)^i \left(e^{-z_1^2/2N_0}\right)^{M-i}$$

(8.17) (Cont.)

and so

$$\Pr\{\text{Correct}/H_1\} = \int_0^\infty \frac{z_1}{E\sigma^2+N_0} e^{-z_1^2/2(E\sigma^2+N_0)} \left[\sum_{i=0}^{M-1} \binom{M-1}{i} e^{-\frac{(M-1-i)}{2N_0} z_1^2}\right] dz_1$$

$$= \sum_{i=0}^{M-1} \frac{\binom{M-1}{i}}{E\sigma^2+N_0} \int_0^\infty z_1 e^{-K_i z_1^2} dz_1$$

where

$$K_i = \frac{1}{2(E\sigma^2+N_0)} + \frac{M-1-i}{2N_0} = \frac{N_0 + (E\sigma^2+N_0)(M-1-i)}{2N_0(E\sigma^2+N_0)}$$

We can write $\int_0^\infty z_1 e^{-K_i z_1^2} dz_1 = \frac{1}{2K_i}$. Thus

$$\Pr\{\text{Correct}/H_1\} = \sum_{i=0}^{M-1} \frac{\binom{M-1}{i}}{E\sigma^2+N_0} \cdot \frac{N_0(E\sigma^2+N_0)}{N_0 + (E\sigma^2+N_0)(M-1-i)}$$

$$= \sum_{i=0}^{M-1} \binom{M-1}{i} \frac{N_0}{N_0 + (E\sigma^2+N_0)(M-1-i)}.$$

We can find $P(\text{Error}/H_1)$ by

$$P(\text{Error}/H_1) = 1 - P(\text{Correct}/H_1).$$

It follows that this expression also yields the unconditional probability of error since we have assumed equal energies and equally likely hypotheses.

(8.18)

$$Y_1 = \sum_{n=1}^{N} R_{1n}^2 \quad ; \quad Y_2 = \sum_{n=1}^{N} R_{2n}^2$$

Given Hi transmitted, the received signal in subpath n is

$$H_1: \quad y_n(t) = \sqrt{\frac{2E}{NT}} \left[G_{1n} \cos \omega_1 t + G_{2n} \sin \omega_1 t \right] + n_n(t)$$

$$H_2: \quad y_n(t) = \sqrt{\frac{2E}{NT}} \left[G_{1n} \cos \omega_2 t + G_{2n} \sin \omega_2 t \right] + n_n(t)$$

where $n_n(t)$ is a Gaussian random process with power spectral density $N_0/2$ and G_{1n}, G_{2n} are independent Gaussian random variables with mean $= 0$ and variance $= \sigma^2$. We express $y_n(t)$ in terms of the four dimensional basis which is

$$\phi_1 = \sqrt{2/T} \cos \omega_1 t \qquad \phi_2 = \sqrt{2/T} \sin \omega_1 t$$

$$\phi_3 = \sqrt{2/T} \cos \omega_2 t \qquad \phi_3 = \sqrt{2/T} \sin \omega_2 t \; .$$

Thus

$$z_{in} = (y_n, \phi_i) = \int_0^T y_n(t) \phi_i(t) \, dt$$

Given H_1:

$$z_{in} = \begin{cases} \sqrt{E/N} \, G_{in} + N_{in} & i = 1, 2 \\ N_{in} & i = 3, 4 \end{cases}$$

and given H_2:

$$z_{in} = \begin{cases} N_{in} & i = 1, 2 \\ \sqrt{E/N} \, G_{in} + N_{in} & i = 3, 4 \end{cases}$$

where $N_{in} = (n_n, \phi_i) = \int_0^T n_n(t) \phi_i(t) \, dt \quad i = 1, 2, 3, 4$

and so $E\{N_{in}\} = 0 \; ; \; \text{Var}\{N_{in}\} = N_0/2 \quad \forall \; i.$

(8.18) (Cont.)

Now for the n^{th} receiver

$$R_{1n}^2 = z_{1n}^2 + z_{2n}^2 \qquad R_{2n}^2 = z_{3n}^2 + z_{4n}^2$$

and so

$$Y_1 = \sum_{n=1}^{N} (z_{1n}^2 + z_{2n}^2) \qquad Y_2 = \sum_{n=1}^{N} (z_{3n}^2 + z_{4n}^2).$$

Now suppose hypothesis H_1 is given. Then

$$E\{z_{1n}\} = E\{z_{2n}\} = 0$$

$$Var\{z_{1n}\} = Var\{z_{2n}\} = \frac{E}{N}\sigma^2 + \frac{N_0}{2}$$

$$E\{z_{3n}\} = E\{z_{4n}\} = 0$$

$$Var\{z_{3n}\} = Var\{z_{4n}\} = N_0/2.$$

Hence Y_1 = sum of squares of $2N$ Gauss random variables with means zero and variances $E/N\sigma^2 + N_0/2$.

Y_2 = sum of squares of $2N$ Gauss random variables with means zero and variances $N_0/2$. Therefore, from problem 4.55, Y_1 and Y_2 are Chi-squared with $2N$ degrees of freedom. Letting N in problem 4.55(b) be $2N$ here, and σ in problem 4.55(b) be $\frac{E}{N}\sigma^2 + \frac{N_0}{2}$ for Y_1 and $N_0/2$ for Y_2, we have

$$f_{Y_1}(y_1/H_1) = \begin{cases} \dfrac{y_1^{N-1}\, e^{-y_1/2(E\sigma^2/N + \frac{N_0}{2})}}{2^N (E\sigma^2/N + N_0/2)^N\, \Gamma(N)} & y_1 \geq 0 \\ 0 & y_1 < 0 \end{cases}$$

$$f_{Y_2}(y_2/H_1) = \begin{cases} \dfrac{y_2^{N-1}\, e^{-y_2/N_0}}{2^N (N_0/2)^N\, \Gamma(N)} & y_2 \geq 0 \\ 0 & y_2 < 0 \end{cases}$$

(8.18) (Cont.)

(b) $P_E = Pr\{E/H_1\} = Pr\{E/H_2\} = Pr\{y_2 > y_1 / H_1\}$

$$= \int_0^\infty \left\{ \int_{y_1}^\infty \frac{y_2^{N-1} e^{-y_2/N_0}}{2^N (N_0/2)^N \Gamma(N)} dy_2 \right\} \frac{y_1^{N-1} e^{-y_1/(2E\sigma^2/N + N_0)}}{2^N (E\sigma^2/N + N_0/2)^N \Gamma(N)} dy_1$$

Clearly, we need an expression for the integral $I_n(x;a) = \int_x^\infty z^n e^{-az} dz$. This is found by integrating by parts.

$u = z^n \quad dv = e^{-az} dz$

$du = n z^{n-1} dz \quad v = -\frac{1}{a} e^{-az}$

Then

$$I_n(x;a) = [uv - \int v\,du]_x^\infty$$

$$= -\frac{z}{a} e^{-az}\Big|_x^\infty + \int_x^\infty \frac{n}{a} z^{n-1} e^{-az} dz$$

$$= \frac{x^n}{a} e^{-ax} + \frac{n}{a} I_{n-1}(x;a)$$

$$= \frac{x^n e^{-ax}}{a} + \frac{n}{a}\left[\frac{x^{n-1} e^{-ax}}{a} + \frac{n-1}{a} I_{n-2} \right]$$

$$= \frac{x^n e^{-ax}}{a} + \frac{n}{a}\left\{ \frac{x^{n-1} e^{-ax}}{a} + \frac{n-1}{a}\left[\frac{x^{n-2} e^{-ax}}{a} + \frac{n-2}{a} I_{n-2} \right] \right\}$$

$$= \frac{x^n e^{-ax}}{a} + \frac{n x^{n-1} e^{-ax}}{a^2} + \frac{n(n-1) x^{n-2} e^{-ax}}{a^3} + \cdots + \frac{n!\, x^0 e^{-ax}}{a^{n+1}}$$

$$= e^{-ax} \sum_{\ell=0}^{n} x^\ell \frac{n!}{\ell!\, a^{n+1-\ell}}$$

where $I_0(x;a) = \frac{1}{a} e^{-ax}$ has been used. So, the inner integral in the expression above for P_E is

$$\frac{I_{N-1}(y_1; 1/N_0)}{2^N (N_0/2)^N \Gamma(N)} = \frac{e^{-y_1/N_0}}{N_0^N} \sum_{\ell=0}^{N-1} y_1^\ell / \ell! \, (1/N_0)^{N-\ell}$$

(8.18) (Cont.)

Therefore

$$P_E = \int_0^\infty \frac{y_1^{N-1} e^{-y_1/(2E\sigma^2/N+N_0)}}{2^N (E\sigma^2/N+\frac{N_0}{2})^N (N-1)!} \cdot \frac{e^{-y_1/N_0}}{N_0^N} \cdot \sum_{\ell=0}^{N-1} \frac{y_1^\ell N_0^{N-\ell}}{\ell!} \, dy_1$$

$$= \sum_{\ell=0}^{N-1} \frac{N_0^{N-\ell}}{(2E\sigma^2/N+\frac{N_0}{2})^N (N-1)! \, N_0^N \, \ell!} \int_0^\infty y_1^{N-1+\ell} e^{-y_1\left[\frac{1}{2E\sigma^2/N+N_0}+\frac{1}{N_0}\right]} dy_1$$

From the expression for $I_n(x;a)$ we have $I_n(0;a) = n!/a^{n+1}$. Thus

$$P_E = \sum_{\ell=0}^{N-1} \frac{N_0^{N-\ell} (N-1+\ell)!}{\left(\frac{2E\sigma^2}{N}+N_0\right)^N (N-1)! \, N_0^N \, \ell! \left[\frac{N}{2E\sigma^2+NN_0}+\frac{1}{N_0}\right]^{N+\ell}}$$

(c) The expression for P_E as written above is difficult to compute. However, it can be rewritten as

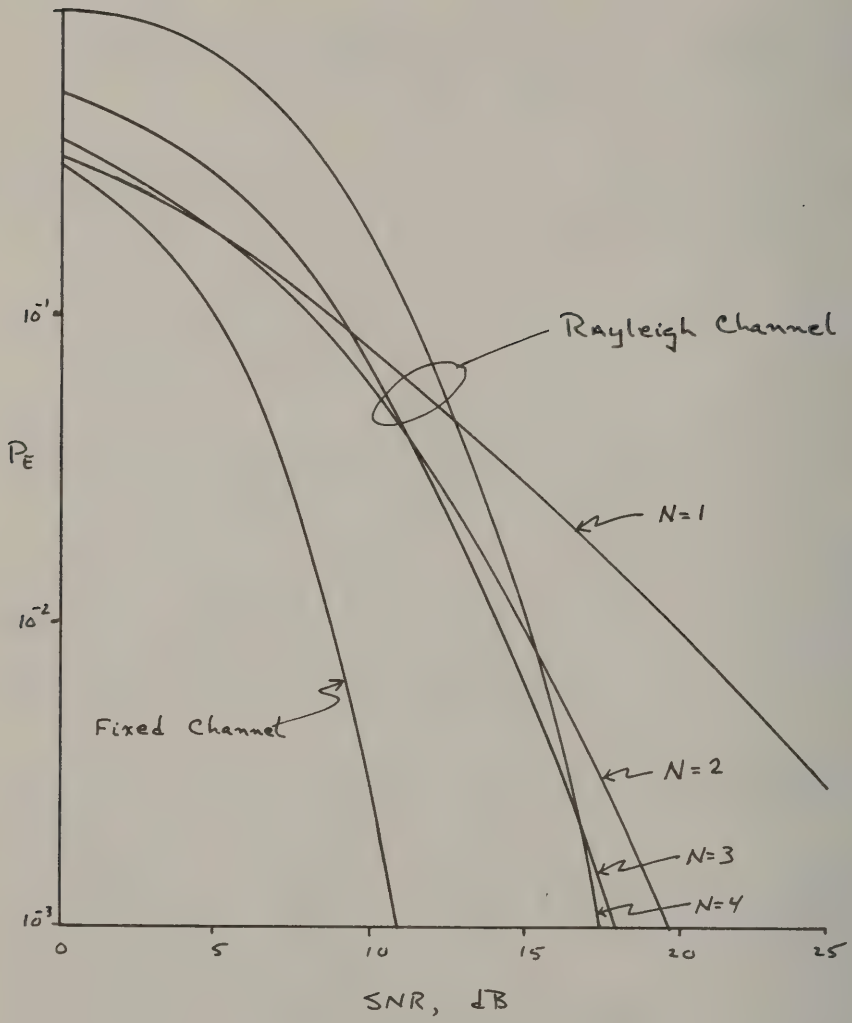
$$P_E = \alpha^N (T_0 + T_1 + \ldots + T_{N-1})$$

where $\alpha = \frac{1}{2+\gamma/N}$ and $\gamma = \frac{2\sigma^2 E}{N_0}$ is the signal-to-noise ratio and where

$$T_{n+1} = \frac{N+n}{n+1}(1-\alpha) T_n \; ; \; T_0 = 1.$$

The plot of P_E for various SNR and number of subpaths, N, is shown below where the scale is that of Fig. 8.10.

8.18 (Cont.)


Rayleigh Channel
N=1
N=2
N=3
N=4
Fixed Channel
P_E
10^{-1}
10^{-2}
10^{-3}
0
5
10
15
20
25
SNR, dB

(8.19)

(a) The characteristic function $\Phi(\omega)$ is found and used to find the desired moments.

$$\Phi(\omega) = E\{e^{j\lambda\omega}\} = \int_0^\infty e^{j\lambda\omega}\frac{\beta^m}{\Gamma(m)}e^{-\beta\lambda}\lambda^{m-1}\,d\lambda.$$

Note that $\int_0^\infty e^{j\lambda\omega}e^{-\beta\lambda}\,d\lambda = \frac{1}{\beta - j\omega}$. (1)

Differentiating both sides of Eqn. (1) $m-1$ times with respect to β gives

$$\int_0^\infty e^{j\lambda\omega}(-1)^{m-1}\lambda^{m-1}e^{-\beta\lambda}\,d\lambda = \frac{(-1)^{m-1}(m-1)!}{(\beta - j\omega)^m}.$$

Multiplying both sides by β^m and cancelling like terms gives

$$\int_0^\infty e^{j\lambda\omega}\frac{\beta^m}{\Gamma(m)}e^{-\beta\lambda}\lambda^{m-1}\,d\lambda = \frac{\beta^m}{(\beta - j\omega)^m}$$

where $\Gamma(m) = (m-1)!$ has been used. Thus

$$\Phi(\omega) = \frac{\beta^m}{(\beta - j\omega)^m}.$$

$$\frac{\partial\Phi(\omega)}{\partial\omega} = \frac{-jm\beta^m}{(\beta - j\omega)^{m+1}} \Rightarrow E\{\lambda\} = j\Phi'(0) = \frac{m}{\beta}.$$

$$\frac{\partial^2\Phi(\omega)}{\partial\omega^2} = \frac{-m(m+1)\beta^m}{(\beta - j\omega)^{m+2}} \Rightarrow E\{\lambda^2\} = j^2\Phi''(0) = m(m+1)/\beta.$$

$$\mathrm{Var}(\lambda) = E\{\lambda^2\} - E^2\{\lambda\} = m/\beta^2.$$

(b)

$$f_{\Lambda/z}(\lambda/z) = \frac{f(z/\lambda)f(\lambda)}{f(z)}.$$

So we need to find $f(z)$.

$$f_z(z) = \int_0^\infty f_{z/\Lambda}(z/\lambda)f(\lambda)\,d\lambda = \int_0^\infty \lambda e^{-\lambda z}\frac{\beta^m}{\Gamma(m)}e^{-\beta\lambda}\lambda^{m-1}\,d\lambda$$

$$= \frac{m\beta^m}{(\beta+z)^{m+1}}\underbrace{\int_0^\infty e^{-(\beta+z)\lambda}\cdot\frac{(\beta+z)^{m+1}}{\Gamma(m+1)}\lambda^m\,d\lambda}$$

This is a Pdf so $\int = 1$.

(8.19) (Cont.)

$$\therefore \quad f_Z(z) = \frac{m\beta^m}{(\beta+z)^{m+1}}$$

It follows that the conditional pdf of λ given z is

$$f_{\Lambda/Z}(\lambda/z) = \frac{\lambda^m (\beta+z)^{m+1} e^{-(\beta+z)\lambda}}{\Gamma(m+1)}$$

Note that this is the same pdf as in part (a) except that $\beta \to \beta+z$, $m \to m+1$.

$$\therefore E\{\Lambda/z\} = \frac{m+1}{\beta+z}, \quad Var\{\Lambda/z\} = \frac{m+1}{(\beta+z)^2}.$$

(c) Assume that z_1, z_2 are conditionally independent. That is,

$$f_{Z_1,Z_2/\Lambda}(z_1,z_2/\lambda) = \begin{cases} \lambda^2 e^{-\lambda(z_1+z_2)}, & z_1>0 \text{ \& } z_2>0. \\ 0, & \text{otherwise} \end{cases}$$

Then

$$f_{Z_1,Z_2}(z_1,z_2) = \int_0^\infty f_{Z_1,Z_2/\Lambda}(z_1,z_2/\lambda) f(\lambda)\, d\lambda$$

$$= \int_0^\infty \lambda^2 e^{-\lambda(z_1+z_2)} \cdot \frac{\beta^m}{\Gamma(m)} e^{-\beta\lambda} \lambda^{m-1}\, d\lambda.$$

Rearranging as before gives

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{m(m+1)\beta^m}{(\beta+z_1+z_2)^{m+2}} \underbrace{\int_0^\infty \lambda^{m+1} e^{-(\beta+z_1+z_2)\lambda} \cdot \frac{(\beta+z_1+z_2)^{m+2}}{\Gamma(m+2)}\, d\lambda}_{\text{This is a pdf so } \int = 1.}$$

$$\therefore f_{Z_1,Z_2}(z_1,z_2) = \frac{m(m+1)\beta^m}{(\beta+z_1+z_2)^{m+2}}$$

(8.19) (Cont.)

and

$$f_{\Lambda/z_1,z_2}(\lambda/3_1,3_2) = \frac{\lambda^{m+1} e^{-\lambda(\beta+3_1+3_2)} (\beta+3_1+3_2)^{m+2}}{\Gamma(m+2)}$$

It follows that

$$E\{\Lambda/z_1,z_2\} = \frac{m+2}{\beta+3_1+3_2}, \quad Var\{\Lambda/z_1,z_2\} = \frac{m+2}{\beta+3_1+3_2}.$$

(d) By inspecting the solution of parts (b) and (c) we can generalize as follows

$$f_{\Lambda \mid z_1,z_2,\ldots,z_K}(\lambda \mid z_1,z_2,\ldots,z_K) = \frac{\lambda^{m-1+K} e^{-\lambda[\beta+\sum_{i=1}^{K} 3_i]} [\beta+\sum_{i=1}^{K} 3_i]^{m+K}}{\Gamma(m+K)}$$

$$E\{\Lambda \mid z_1,z_2,\ldots,z_K\} = \frac{m+K}{[\beta+\sum_{i=1}^{K} 3_i]}$$

$$Var\{\Lambda \mid z_1,z_2,\ldots,z_K\} = \frac{m+K}{[\beta+\sum_{i=1}^{K} 3_i]^2}$$

(8.20)

The conditional mean & Bayes' estimate are the same if:

1) $C(x)$ is symmetric
2) $C(x)$ is convex-upward
3) $P(a/3)$ is symmetric about mean.

(a) Same. All conditions satisfied.

(b) Same. All conditions satisfied

(c) Condition 3 not satisfied.

(d) Condition 2 not satisfied.

(8.21)

(a) The conditional pdf of the noise samples is

$f_{\underline{z}}(z_1, z_2, \ldots, z_K / \sigma_N^2) = \left(\sqrt{2\pi\sigma_N^2}\right)^{-K} \exp\left(-\frac{1}{2\sigma_N^2}\sum_{i=1}^{K} z_i^2\right)$.

The maximum likelihood estimate $\hat{\sigma}_N^2$ is that σ_N^2 that maximizes $f_{\underline{z}}(z_1, z_2, \ldots, z_K / \sigma_N^2)$.

Differentiating with respect to σ_N^2 gives

$$\frac{\partial f_{\underline{z}}}{\partial \sigma_N^2} = (2\pi)^{-K/2}\left[-\frac{K}{2}(\sigma_N^2)^{-K/2-1} + \frac{1}{2}(\sigma_N^2)^{-K/2-2}\sum_{i=1}^{K} z_i^2\right] e^{-\frac{1}{2\sigma_N^2}\sum_{i=1}^{K} z_i^2}$$

$$= 0 \Rightarrow \frac{1}{\sigma_N^2}\sum_{i=1}^{K} z_i^2 = K \Rightarrow \hat{\sigma}_N^2 = \frac{1}{K}\sum_{i=1}^{K} z_i^2.$$

(b) $E\{\hat{\sigma}_N^2\} = \frac{K}{K}\sigma_N^2 = \sigma_N^2$.

$$E\{(\hat{\sigma}_N^2)^2\} = \frac{1}{K^2} E\left\{\left[\sum_{i=1}^{K} z_i^2\right]\left[\sum_{j=1}^{K} z_j^2\right]\right\}$$

$$= \frac{1}{K^2} K E\{z_i^4\} + \frac{K^2-K}{K^2} E\{z_i^2 z_j^2\} \qquad i \neq j$$

$$= K^{-1} E\{z_i^4\} + (1-K^{-1}) E\{z_i^2 z_j^2\} \qquad i \neq j$$

$$= 3K^{-1}\sigma_N^4 + (1-K^{-1})\sigma_N^4 = \sigma_N^4 + 2K^{-1}\sigma_N^4.$$

$\therefore$ $\mathrm{Var}\{\hat{\sigma}_N^2\} = 2K^{-1}\sigma_N^4$.

(c) Yes, $\hat{\sigma}_N^2$ is unbiased.

(d) $\sum_{i=1}^{K} z_i^2$ is a sufficient statistic.

(8.22)

From (8-97), we base our estimate upon the observation of z_0 where $z_0 = M_0 + N$, and N is a zero mean Gaussian random variable with variance $N_0/2$. The sample value, M_0, is assumed zero-mean Gaussian with variance σ_m^2. The figure of merit is mean-squared error. Thus we use

(8.22) (Cont.)

a maximum a priori (MAP) estimate. It can be shown that since our processes are Gaussian, that the estimate $\hat{m}_0$ is the mean value of M_0 conditioned on the observation z_0. That is,

$$\hat{M}_0 = E\{M_0 / z_0 = z_0\}.$$

It is easily shown (see Example 8.8) that this conditional mean is simply

$$E\{M_0 / z_0 = z_0\} = \rho z_0$$

where ρ is given by

$$\rho = \frac{E\{M_0 z_0\}}{[\mathrm{Var}(M_0)]^{1/2}[\mathrm{Var}(z_0)]^{1/2}} = \frac{\sigma_m}{[\sigma_m^2 + N_0/2]^{1/2}}$$

$$\therefore \quad \hat{M}_{0,\mathrm{MAP}} = \frac{\sigma_m}{[\sigma_m^2 + N_0/2]^{1/2}} z_0.$$

It is of interest to compare this estimate to the maximum-likelihood (ML) estimate. The ML estimate is

$$\hat{M}_{0,\mathrm{ML}} = z_0.$$

This is the case when we have no a priori knowledge. Note that the MAP estimate reduces to this when $\sigma_m^2 \to \infty$ (i.e., we have no a priori knowledge). Note also that for $N_0/2 \to \infty$ (i.e., small SNR) we don't use the observed value of z_0, but instead estimate $\hat{M}_0$ as zero.

(8.23) The received signal is

H_1: $y(t) = A\cos(\omega_0 t + \theta) + n(t)$, $0 \le t \le T$

H_2: $y(t) = -A\cos(\omega_0 t + \theta) + n(t)$, $0 \le t \le T$

where A has been introduced so that we can easily the SNR later on. The signal $y(t)$ can be expanded to give

H_1: $y(t) = A\cos\omega_0 t\cos\theta - A\sin\omega_0 t\sin\theta + n(t)$

H_2: $y(t) = -A\cos\omega_0 t\cos\theta + A\sin\omega_0 t\sin\theta + n(t)$

We can express $y(t)$ in terms of the basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos\omega_0 t \qquad \phi_2(t) = \sqrt{\frac{2}{T}}\sin\omega_0 t$$

so that in vector notation we have

H_1: $z = (z_1, z_2) = \left(\sqrt{\frac{T}{2}}A\cos\theta + N_1, -\sqrt{\frac{T}{2}}A\sin\theta + N_2\right)$

H_2: $z = (z_1, z_2) = \left(-\sqrt{\frac{T}{2}}A\cos\theta + N_1, \sqrt{\frac{T}{2}}A\sin\theta + N_2\right)$

where

$$N_1 = \int_0^T n(t)\phi_1(t)\,dt \quad ; \quad N_2 = \int_0^T n(t)\phi_2(t)\,dt$$

and

$$E\{N_1\} = E\{N_2\} = 0$$

$$\text{var}\{N_1\} = \text{var}\{N_2\} = \int_0^T\int_0^T R_n(t-\sigma)\phi_1(t)\phi_2(\sigma)\,dt\,d\sigma$$

$$= \frac{1}{2}N_0\int_0^T \phi_1^2(t)\,dt = \frac{1}{2}N_0$$

Thus

$$E\{z_1|H_1\} = \sqrt{\frac{T}{2}}A\cos\theta \triangleq A_1$$

$$E\{z_2|H_1\} = -\sqrt{\frac{T}{2}}A\sin\theta \triangleq -A_2$$

$$E\{z_1|H_2\} = -\sqrt{\frac{T}{2}}A\cos\theta \triangleq -A_1$$

$$E\{z_2|H_2\} = \sqrt{\frac{T}{2}}A\sin\theta \triangleq A_2$$

and

$$\text{var}\{N_i|H_j\} = \frac{N_0}{2}, \qquad i, j = 1, 2.$$

(8.23) (Cont)

Therefore

$$f_{z|\theta,H_1}(z_1,z_2|\theta,H_1) = \frac{1}{\pi N_0}\exp\left\{-\frac{1}{N_0}\left[(z_1-A_1)^2+(z_2+A_2)^2\right]\right\}$$

$$f_{z|\theta,H_2}(z_1,z_2|\theta,H_2) = \frac{1}{\pi N_0}\exp\left\{-\frac{1}{N_0}\left[(z_1+A_1)^2+(z_2-A_2)^2\right]\right\}$$

(b) we assume the hypotheses equally likely so that $P(H_1) = P(H_2) = \frac{1}{2}$. Then

$$f_{z|\theta}(z_1,z_2|\theta) = \frac{1}{2\pi N_0}\left\{\exp\left[-\frac{1}{N_0}\left((z_1-A_1)^2+(z_2+A_2)^2\right)\right] + \exp\left[-\frac{1}{N_0}\left((z_1+A_1)^2+(z_2-A_2)^2\right)\right]\right\}$$

$$= \frac{1}{2\pi N_0}\left\{\exp\left[\frac{-1}{N_0}\left(z_1^2-2z_1A_1+A_1^2+z_2^2+2A_2z_2+A_2^2\right)\right] + \exp\left[\frac{-1}{N_0}\left(z_1^2+2z_1A_1+A_1^2+z_2^2-2A_2z_2+A_2^2\right)\right]\right\}$$

we note that

$$A_1^2+A_2^2 = \frac{\Gamma}{2}A^2\cos^2\theta + \frac{\Gamma}{2}A^2\sin^2\theta = \frac{\Gamma}{2}A^2$$

so that we can write

$$f_{z|\theta}(z_1,z_2|\theta) = \frac{1}{2\pi N_0}\exp\left[\frac{-1}{N_0}\left(z_1^2+z_2^2+\frac{\Gamma}{2}A^2\right)\right] \cdot \left\{\exp\left[\frac{+2}{N_0}\left(A_1z_1-A_2z_2\right)\right]+\exp\left[\frac{-2}{N_0}\left(A_1z_1-A_2z_2\right)\right]\right\}$$

Thus

$$f_{z|\theta}(z_1,z_2|\theta) = \frac{1}{\pi N_0}\exp\left[\frac{-1}{N_0}\left(z_1^2+z_2^2+\frac{\Gamma}{2}A^2\right)\right]\cosh\left[\frac{2}{N_0}\left(A_1z_1-A_2z_2\right)\right]$$

The maximum likelihood estimate for θ can be obtained from

$$\frac{\partial}{\partial\theta}\ln f_{z|\theta}(z_1,z_2|\theta)\Big|_{\theta=\hat{\theta}_{ML}} = 0$$

Thus

$$\ln f_{z|\theta}(z_1,z_2|\theta) = \frac{-1}{N_0}\left(z_1^2+z_2^2+\frac{\Gamma}{2}A^2\right) + \ln\cosh\left[\sqrt{\frac{\Gamma}{2}}\frac{2A}{N_0}\left(z_1\cos\theta - z_2\sin\theta\right)\right]$$

(8.23) (Cont)

$$\frac{\partial}{\partial\theta}\ln f_{z|\theta}(z_1,z_2|\theta) = 0 + \tanh\left[\sqrt{\frac{T}{2}}\frac{2A}{N_0}(z_1\cos\theta - z_2\sin\theta)\right] \cdot \sqrt{\frac{T}{2}}\frac{2A}{N_0}(-z_1\sin\theta - z_2\cos\theta)$$

Therefore the condition for $\theta = \hat{\theta}_{ML}$ is

$$\tanh\left[\sqrt{\frac{T}{2}}\frac{2A}{N_0}(z_1\cos\theta - z_2\sin\theta)\right] \cdot \sqrt{\frac{T}{2}}\frac{2A}{N_0}(-z_1\sin\theta - z_2\cos\theta)\Big|_{\theta=\hat{\theta}_{ML}} = 0$$

Now

$$z_1 = \int_0^T y(t)\sqrt{\frac{2}{T}}\cos\omega_0 t\,dt; \quad z_2 = \int_0^T y(t)\sqrt{\frac{2}{T}}\sin\omega_0 t\,dt$$

so that

$$z_1\cos\theta - z_2\sin\theta = \sqrt{\frac{2}{T}}\int_0^T y(t)\cos(\omega_0 t+\theta)\,dt$$

$$-z_1\sin\theta - z_2\cos\theta = \sqrt{\frac{2}{T}}\int_0^T y(t)\sin(\omega_0 t+\theta)\,dt$$

and so $\hat{\theta}_{ML}$ satisfies the equation

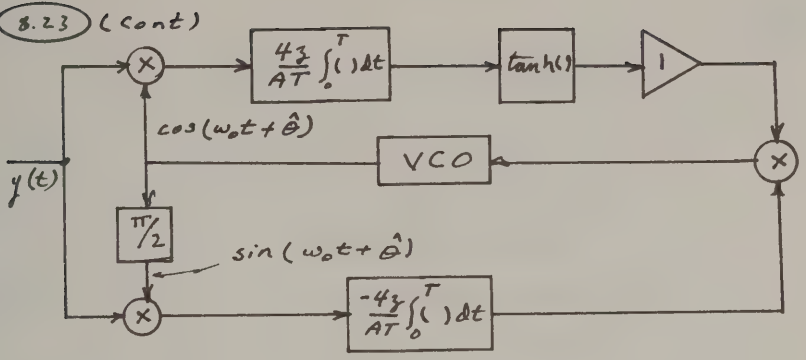
$$\tanh\left[\frac{4\gamma}{AT}\int_0^T y(t)\cos(\omega_0 t+\theta)\,dt\right]\frac{4\gamma}{AT}\int_0^T -y(t)\sin(\omega_0 t+\theta)\,dt = 0$$

where $\gamma = \frac{A^2T}{2N_0}$ is the signal-to-noise ratio. This is realized by the structure shown at the top of the following page. Note that for SNR = γ small, the argument of $\tanh(x)$ is small and $\tanh(x) \approx x$. In this case the loop acts like a Costas PLL except for integrators instead of lowpass filters. Recall, however, that an integrator is approximated by a lowpass filter.

(c) To find an expression for $\mathrm{var}\{\hat{\theta}_{ML}\}$ we evaluate the Cramer-Rao bound, which is

$$\mathrm{var}\{\hat{\theta}_{ML}\} \geq E\left\{\frac{\partial^2}{\partial\theta^2}\ln f_{z|\theta}\right\}^{-1}$$

8.23 (cont)



First we compute

$$\frac{\partial}{\partial\theta}\ln f_{z|\theta} = \frac{2A}{N_0}\tanh\left[\frac{2A}{N_0}\int_0^T y(t)\cos(\omega_0 t+\theta)\,dt\right] \cdot \int_0^T -y(t)\sin(\omega_0 t+\theta)\,dt$$

Assume a low signal-to-noise ratio. Then $\tanh(x) \approx x$ and

$$\frac{\partial}{\partial\theta}\ln f_{z|\theta} = \left(\frac{2A}{N_0}\right)^2\int_0^T y(t)\cos(\omega_0 t+\theta)\,dt\int_0^T -y(t)\sin(\omega_0 t+\theta)\,dt$$

$$\frac{\partial^2}{\partial\theta^2}\ln f_{z|\theta} = \left(\frac{2A}{N_0}\right)^2\left\{\left[\int_0^T y(t)\sin(\omega_0 t+\theta)\,dt\right]^2 - \left[\int_0^T y(t)\cos(\omega_0 t+\theta)\,dt\right]^2\right\}$$

$$= \left(\frac{2A}{N_0}\right)^2\left\{\int_0^T\int_0^T y(t)\,y(t')\sin(\omega_0 t+\theta)\sin(\omega_0 t'+\theta)\,dt\,dt' - \int_0^T\int_0^T y(t)\,y(t')\cos(\omega_0 t+\theta)\cos(\omega_0 t'+\theta)\,dt\,dt'\right\}$$

Applying trig identies yields

$$\frac{\partial^2}{\partial\theta^2} f_{z|\theta} = \left(\frac{2A}{N_0}\right)^2\int_0^T\int_0^T y(t)\,y(t')\cos\omega_0(t-t')\,dt\,dt'$$

$$E\left\{\frac{\partial^2}{\partial\theta^2} f_{z|\theta}\right\} = \left(\frac{2A}{N_0}\right)^2\int_0^T\int_0^T E\{y(t)\,y(t')\}\cos\omega_0(t-t')\,dt\,dt'$$

We therefore must evaluate $E\{y(t)\,y(t')\}$

$$E\{y(t)\,y(t')\} = E\{y(t)\,y(t')\,|\,H_1\}\,P(H_1) + E\{y(t)\,y(t')\,|\,H_2\}\,P(H_2)$$

(8.23) (cont.)

$$E\{y(t)y(t')\} = \frac{E}{2}\{A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t'+\theta) + n(t)n(t') + A\cos(\omega_0 t+\theta)n(t') + A\cos(\omega_0 t'+\theta)n(t)\}$$
$$+ \frac{E}{2}\{A^2\cos(\omega_0 t'+\theta)\cos(\omega_0 t+\theta) + n(t')n(t) - A\cos(\omega_0 t'+\theta)n(t) - A\cos(\omega_0 t+\theta)n(t')\}$$
$$= E\{A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t'+\theta) + n(t)n(t')\}$$
$$= E\{A^2[\tfrac{1}{2}\cos(\omega_0 t+\omega_0 t'+2\theta) + \tfrac{1}{2}\cos\omega_0(t-t')]\} + \frac{N_0}{2}\delta(t-t')$$

Thus

$$E\{y(t)y(t')\} = \tfrac{1}{2}A^2\cos\omega_0(t-t') + \frac{N_0}{2}\delta(t-t')$$

Therefore

$$E\left\{\frac{\partial^2 \ln f_{z|\theta}}{\partial\theta^2}\right\} = \left(\frac{2A}{N_0}\right)^2\left\{\int_0^T\int_0^T \frac{A^2}{2}\cos^2\omega_0(t-t')\,dt\,dt' + \tfrac{1}{2}N_0T\right\}$$
$$= \left(\frac{2A}{N_0}\right)^2\left[\frac{A^2T^2}{4} + \frac{N_0T}{2}\right]$$

Since the SNR γ is $\frac{A^2T}{2N_0}$ we have

$$E\left\{\frac{\partial^2 f_{z|\theta}}{\partial\theta^2}\right\} = \frac{4A^2}{N_0^2}\left(\frac{A^2T^2}{4}\right) + \frac{4A^2T}{2N_0}$$
$$= 4\gamma + 4\gamma^2 = 4\gamma(\gamma+1)$$

So finally

$$\text{var}\{\hat{\theta}_{ML}\} \geq \frac{1}{4\gamma(\gamma+1)}$$

From Table 7.5 the phase-error variance with PSK and a Costas loop is

$$\sigma_\theta^2 = \frac{1}{Q}\left(\frac{1}{\gamma} + \frac{1}{2\gamma^2}\right), \quad Q = 1/B_L T_3$$

(8.24)

We want to evaluate $I = \int_0^T \left(\frac{\partial s(t,A)}{\partial A}\right)^2 dt$, where

$$s(t,A) = B\phi(t - AT_0 - T/2)$$

and $\phi(t) = \sqrt{2W}\,\mathrm{sinc}\,2Wt$.

$$s(t,A) = B\sqrt{2W}\,\mathrm{sinc}\,2W(t - AT_0 - T/2).$$

Since $T_0 \ll T$ and $|A| \le 1$, the waveform $s(t,A)$, although not strictly time-limited because it is bandlimited, has nearly all its energy in the T-sec. interval. Furthermore, we assume the waveform $\frac{\partial}{\partial A} s(t,A)$ has nearly all its energy in the T-sec interval. This being the case we can use Parseval's theorem to evaluate I. From the appendix,

$$s(t,A) \Longleftrightarrow \frac{B}{\sqrt{2W}}\,\Pi(f/2W)\,e^{-j2\pi f(AT_0 + T/2)}$$

Applying the differentiation theorem gives

$$\frac{\partial s(t,A)}{\partial A} \Longleftrightarrow j2\pi f T_0\, S(f).$$

To see this write $\frac{\partial s(t,A)}{\partial A} = \frac{\partial}{\partial A}\mathcal{F}^{-1}\{\mathcal{F}[s(t,A)]\}$ and note A appears only as a complex exponential in $\mathcal{F}\{s(t,A)\}$. Thus

$$I = \int_0^T \left(\frac{\partial s(t,A)}{\partial A}\right)^2 dt \approx \int_{-\infty}^{\infty} \left(\frac{\partial s(t,A)}{\partial A}\right)^2 dt$$

$$= \int_{-\infty}^{\infty} \left|\mathcal{F}\left\{\frac{\partial s(t,A)}{\partial A}\right\}\right|^2 df = \int_{-\infty}^{\infty} \frac{B^2}{2W}(2\pi f T_0)^2\,\Pi(f/2W)\,df$$

$$= 2(BT_0\pi)^2/W \int_W^W f^2\,df = (2\pi B T_0 W)^2/3.$$

(8.25)

Equation (8.130) indicates to choose $\hat{A}_{ML}$ so that

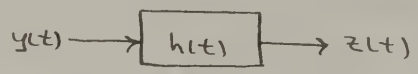
$$\frac{2}{N_0}\int_0^T y(t)\, B\, \phi(t - A_0 T_0 - T/2)\, dt$$

is a maximum, where

$$y(t) = s(t, A) + n(t)$$

$$s(t, A) = \phi(t - A\, T_0 - T/2).$$

Suppose we pass $y(t)$ thru a filter with impulse response $h(t)$ as shown.



Then the output $z(t)$ is given by

$$z(t) = \int_{-\infty}^{\infty} y(\tau)\, h(t - \tau)\, d\tau.$$

Now suppose $h(t)$ is given by

$$h(t) = \phi(-t - T/2).$$

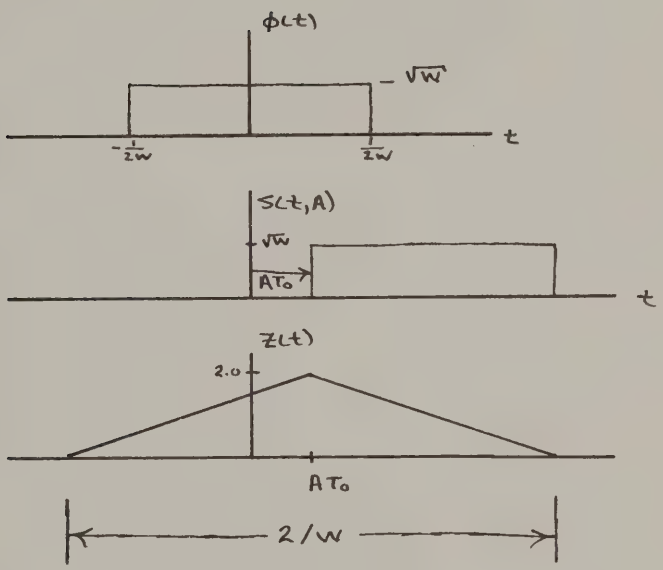
Then

$$z(t) = \int_{-\infty}^{\infty} y(\tau)\, \phi(\tau - T/2 - t)\, d\tau.$$

Now comparing this to (8.130) shows that $\hat{A}_{ML}$ is $\hat{t}/T_0$ where $\hat{t}$ is that t for which $z(t)$ is a maximum. (Note that the limits on the convolution integral can be $[0, T]$ instead of $[-\infty, \infty]$ and also $\hat{t} \in [0, T]$. This is due to the range of $y(t)$.)

(8.25) (Cont.)

(b) The various waveforms are shown below.



(c) $z(t) = \int_{-\infty}^{\infty} \phi(\tau - AT_0 - T/2)\, \phi(\tau - T/2 - t)\, d\tau$

$= \int_{-\infty}^{\infty} 2W\pi[2W(\tau - AT_0 - T/2)] \cos\omega_0(\tau - AT_0 - T/2)$

$\cdot \pi[2W(\tau - T/2 - t)] \cos\omega_0(\tau - T/2 - t)\, d\tau$

Using change of variables with $\tau \to \tau + T/2$ gives

$z(t) = \int_{-\infty}^{\infty} 2W\pi[2W(\tau - AT_0)] \cos\omega_0(\tau - AT_0)$

$\cdot \pi[2W(\tau - t)] \cos\omega_0(\tau - t)\, d\tau.$

Assume $A \geq 0$. ($A < 0$ follows by symmetry.) Then

$$z(t) = \begin{cases} \int_{-\frac{1}{W}+AT_0}^{\frac{1}{W}+t} 2W \cos\omega_0(\tau - AT_0) \cos\omega_0(\tau - t)\, d\tau & t < AT_0 \\ \int_{-\frac{1}{W}+t}^{\frac{1}{W}+AT_0} 2W \cos\omega_0(\tau - AT_0) \cos\omega_0(\tau - t)\, d\tau & t \geq AT_0 \end{cases}$$

(8.25) (Cont.)

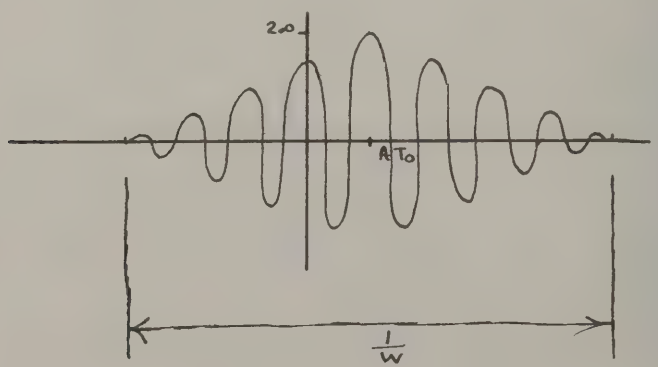
This expression reduces to

$$z(t) = \begin{cases} \int_{-\frac{1}{W}+AT_0}^{\frac{1}{W}+t} W \cos \omega_0 (AT_0 - t)\, d\tau & t < AT_0 \\ \int_{-\frac{1}{W}+t}^{\frac{1}{W}+AT_0} W \cos \omega_0 (AT_0 - t)\, d\tau & t \geq AT_0 \end{cases}$$

where the double frequency term has been assumed zero. Therefore

$$z(t) = \begin{cases} [2 + W(t - AT_0)] \cos \omega_0 (AT_0 - t) & t < AT_0 \\ [2 + W(AT_0 - t)] \cos \omega_0 (AT_0 - t) & t \geq AT_0 \end{cases}$$

Note also that this is true only for $|t - AT_0| \leq \frac{1}{2W}$ since we're convolving two waveforms of duration $\frac{1}{2W}$. The plot is shown below.



(8.25) (Cont.)

(d) Performance is determined by the noise power in $z(t)$ (noise power increases with bandwidth), and by the rate of change of $z(t)$ in the vicinity of $t = AT_0$ (better performance as $\frac{\partial z(t)}{\partial A}$ increases). However, these two criterion are conflicting and so it is difficult to say very much about performance without a detailed analysis. Basically, $\frac{\partial z(t)}{\partial A}$ is much larger for the second waveform so we expect considerably better performance for high SNR. But if we are near threshold, the first waveform will provide better performance since the probability of anomaly here is less than with second waveform.

(8.26)

(a) $s(t) = y(t) - n(t) = \sqrt{2P}\sin(\omega_0 t \pm \cos^{-1} m + \theta) \quad 0 \le t \le T$

$= \sqrt{2P}\sin(\omega_0 t + \theta)\cos(\cos^{-1} m) \pm \sqrt{2P}\cos(\omega_0 t + \theta)\sin(\cos^{-1} m)$

$= \sqrt{2P}\, m \sin(\omega_0 t + \theta) \pm \sqrt{2P}\sqrt{1-m^2}\cos(\omega_0 t + \theta)$

The basis functions of (8-102) are

$$\phi_1(t) = \sqrt{2/T}\cos\omega_0 t \;, \quad \phi_2(t) = \sqrt{2/T}\sin\omega_0 t \qquad 0 \le t \le T$$

Applying trig identities to $s(t)$ gives

$$s(t) = \sqrt{PT}\,(m\sin\theta \pm \sqrt{1-m^2}\cos\theta)\,\phi_1(t) + \sqrt{PT}\,(m\cos\theta \mp \sqrt{1-m^2}\sin\theta)\,\phi_2(t)$$

(b) From part (a) we can base decision on the vector

$$\bar{y} = (y_1, y_2) = \left[\sqrt{PT}\,(m\sin\theta \pm \sqrt{1-m^2}\cos\theta) + N_1 \;,\; \sqrt{PT}\,(m\cos\theta \mp \sqrt{1-m^2}\sin\theta) + N_2\right]$$

$$\triangleq (S_1 + N_1 \;,\; S_2 + N_2)$$

where N_1, N_2 are independent gaussian random variables with variance N_0. We can write the pdf of $\bar{y}$ condition on the signal (+ or −) and θ as

$$f_{\bar{y}/\theta,\pm}(y_1, y_2/\theta, \pm) = \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left[(y_1 - S_1)^2 + (y_2 - S_2)^2\right]}$$

We need to find $f_{\bar{y}/\theta}(y_1, y_2/\theta)$, the pdf of $\bar{y}$ conditioned only on θ. This can be written

$$f_{\bar{y}/\theta}(y_1, y_2/\theta) = f_{\bar{y}/\theta,\pm}(y_1, y_2/\theta, \pm)\Big|_+ P_r\{+\} + f_{\bar{y}/\theta,\pm}(y_1, y_2/\theta, \pm)\Big|_- P_r\{-\}$$

$$= \tfrac{1}{2} f_{\bar{y}/\theta,\pm}(y_1, y_2/\theta, \pm)\Big|_+ + \tfrac{1}{2} f_{\bar{y}/\theta,\pm}(y_1, y_2/\theta, \pm)\Big|_-$$

(8.26) (Cont.)

$$= \frac{1}{\pi N_0} e^{-\frac{1}{N_0}[(y_1 - s_{1,+})^2 + (y_2 - s_{2,+})^2]} + \frac{1}{\pi N_0} e^{-\frac{1}{N_0}[(y_1 - s_{1,-})^2 + (y_2 - s_{2,-})^2]}$$

where the second subscript on s_i denotes the data value. After simplifying we have

$$f_{\bar{Y}/\Theta}(y_1, y_2/\theta) = \frac{1}{\pi N_0} e^{-\frac{1}{N_0}(y_1^2 + y_2^2)} \left[e^{-\frac{1}{N_0}(s_{1,+}^2 + s_{2,+}^2)} e^{-\frac{1}{N_0}(-2y_1 s_{1,+} - 2y_2 s_{2,+})} + e^{-\frac{1}{N_0}(s_{1,-}^2 + s_{2,-}^2)} e^{-\frac{1}{N_0}(-2y_1 s_{1,-} - 2y_2 s_{2,-})} \right]$$

Note that $e^{-\frac{1}{N_0}(y_1^2 + y_2^2)}$ is independent of Θ. Also, the exponentials in $s_{1,\pm}^2 + s_{2,\pm}^2$ are independent of Θ and equal for data + or −. Therefore after considerable simplification we can write

$$f_{\bar{Y}/\Theta}(y_1, y_2/\theta) = C_1 e^{\left[\frac{y_1 2\sqrt{PT}}{N_0} m \sin\theta + \frac{y_2 2\sqrt{PT}}{N_0} m \cos\theta\right]} \cdot \left\{ e^{\left[\frac{y_1 2\sqrt{PT}}{N_0}\sqrt{1-m^2}\cos\theta - \frac{y_2 2\sqrt{PT}}{N_0}\sqrt{1-m^2}\sin\theta\right]} + e^{\left[-\frac{y_1 2\sqrt{PT}}{N_0}\sqrt{1-m^2}\cos\theta + \frac{y_2 2\sqrt{PT}}{N_0}\sqrt{1-m^2}\sin\theta\right]} \right\}$$

$$= C_2 e^{\left[\frac{2m\sqrt{PT}}{N_0}(y_1 \sin\theta + y_2 \cos\theta)\right]} \cosh\left[\frac{2\sqrt{1-m^2}\sqrt{PT}}{N_0}(y_1 \cos\theta - y_2 \sin\theta)\right]$$

where C_1, C_2 are independent of Θ. Taking log of this expression gives $L(\theta)$ where $\log C_2$ has been suppressed. Thus

$$L(\theta) = \frac{2m\sqrt{PT}}{N_0}(y_1 \sin\theta + y_2 \cos\theta) + \ln\cosh\left\{\frac{2\sqrt{1-m^2}\sqrt{PT}}{N_0}(y_1 \cos\theta - y_2 \sin\theta)\right\}$$

We note that

$$y_1 \sin\theta + y_2 \cos\theta = \int_0^T y(t)\sqrt{\tfrac{2}{T}}\sin(\omega_0 t + \theta)\,dt$$

$$y_1 \cos\theta - y_2 \sin\theta = \int_0^T y(t)\sqrt{\tfrac{2}{T}}\cos(\omega_0 t + \theta)\,dt$$

and so

$$L(\theta) = \frac{2m\sqrt{2PT}}{N_0}\int_0^T y(t)\sin(\omega_0 t + \theta)\,dt + \ln\cosh\left\{\frac{\sqrt{1-m^2}\sqrt{2PT}}{N_0}\int_0^T y(t)\cos(\omega_0 t + \theta)\,dt\right\}$$

8.26 (cont.)

(c)

$G_1 = 2\sqrt{2P(1-m^2)}/N_0$

$G_2 = \sqrt{1-m^2}/m$

$X_i(t)$ denotes i^{th} transmission.

(8.27)

Since $R_{ss}(\tau) = A\, e^{-|\tau|/\tau_0}$

$$S_{ss}(f) = \frac{2A\tau_0}{1+(2\pi f \tau_0)^2}$$

Also, since $R_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$, $S_{nn}(f) = \frac{N_0}{2}$

Because of the independence of $s(t)$ and $n(t)$ we have

$$R_{xx}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$$
$$S_{xx}(f) = S_{ss}(f) + S_{nn}(f)$$

The cross correlation and cross PSD is

$$R_{sx}(\tau) = E\{s(t)[s(t+\tau)+n(t+\tau)]\}$$
$$= E\{s(t)s(t+\tau)\} + E\{s(t)n(t+\tau)\}$$

Thus

$$R_{sx}(\tau) = R_{ss}(\tau)$$

and

$$S_{sx}(f) = S_{ss}(f)$$

Therefore, from page 492

$$H_o(f) = \frac{S_{sx}(f)}{S_{xx}(f)} = \frac{S_{ss}(f)}{S_{ss}(f)+S_{nn}(f)}$$

$$= \frac{\frac{2A\tau_0}{1+(2\pi f\tau_0)^2}}{\frac{2A\tau_0}{1+(2\pi f\tau_0)^2} + \frac{N_0}{2}}$$

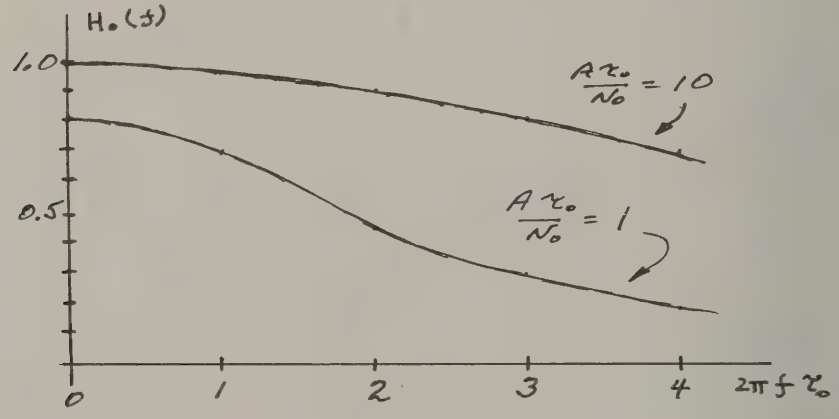
$$= \frac{2A\tau_0}{2A\tau_0 + \frac{N_0}{2}\left[1+(2\pi f\tau_0)^2\right]}$$

(8.27) (cont)

We finally have

$$H_o(f) = \frac{\frac{4A\tau_o}{N_o}}{\frac{4A\tau_o}{N_o} + 1 + (2\pi f \tau_o)^2}$$

$H_o(f)$ is plotted below as a function of $2\pi f \tau_o$ for $\frac{A\tau_o}{N_o} = 1$ and $\frac{A\tau_o}{N_o} = 10$



With $\frac{A\tau_o}{N_o}$ large, which is the low-noise case, the optimum filter $H_o(f)$ is wideband so that all of the signal is passed. For $\frac{A\tau_o}{N_o}$ small, the filter is narrowband compared to the large $\frac{A\tau_o}{N_o}$ case. As N_o increases, the transfer function of the optimum filter, $H_o(f)$, approaches $(\frac{2}{N_o}) S'_{ss}(f)$, which is

(8.27) (cont)

an attenuated version of the signal PSD.

The minimum mean-square error is

$$e_{min} = \int_{-\infty}^{\infty} \frac{S_{ss}(f) S_{nn}(f)}{S_{ss}(f) + S_{nn}(f)} \, df$$

$$= \int_{-\infty}^{\infty} S_{nn}(f) H_o(f) \, df$$

$$= \frac{N_o}{2} \int_{-\infty}^{\infty} \frac{4A\tau_o / N_o}{\frac{4A\tau_o}{N_o} + 1 + (2\pi f \tau_o)^2} \, df$$

$$= \int_{-\infty}^{\infty} \frac{2A\tau_o}{\frac{4A\tau_o}{N_o} + 1 + (2\pi f \tau_o)^2} \, df$$

$$= \frac{2A\tau_o}{\sqrt{1 + \frac{4A\tau_o}{N_o}}} \int_{-\infty}^{\infty} \frac{df \Big/ \sqrt{1 + \frac{4A\tau_o}{N_o}}}{1 + \left(\frac{2\pi f \tau_o}{\sqrt{1 + \frac{4A\tau_o}{N_o}}} \right)^2}$$

Note that $\sqrt{1 + \frac{4A\tau_o}{N_o}} = K$, a constant and that $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.

$$e_{min} = \frac{2A\tau_o}{K^2} \, 2 \int_0^{\infty} \frac{df}{1 + \left(\frac{2\pi f \tau_o}{K} \right)^2}$$

With $x = \frac{2\pi f \tau_o}{K}$ we have

$$e_{min} = \frac{4A\tau_o}{K^2} \int_0^{\infty} \frac{\frac{K}{2\pi\tau_o} \, dx}{1 + x^2}$$

$$= \frac{4A\tau_o}{K^2} \cdot \frac{K}{2\pi\tau_o} \cdot \frac{\pi}{2} = \frac{A}{K}$$

(8.27) (cont)

Thus

$$e_{min} = \frac{A}{\sqrt{1 + \frac{4A\tau_0}{N_0}}}$$

which shows that e_{min} decreases as N_0 decreases; an expected result. As $N_0 \to 0$, $e_{min} \to 0$. Also, as $N_0 \to \infty$, $e_{min} \to A$.

(8.28)

with $R_{ss}(\tau) = A e^{-|\tau|/\tau_0} \cos 2\pi f_0 \tau$

we have

$$S_{ss}(f) = \bar{X}(f) * \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

where, from the previous problem

$$X(f) = \frac{2A\tau_0}{1 + (2\pi f \tau_0)^2}$$

Thus,

$$S_{ss}(f) = \frac{A\tau_0}{1 + [2\pi(f - f_0)\tau_0]^2} + \frac{A\tau_0}{1 + [2\pi(f + f_0)\tau_0]^2}$$

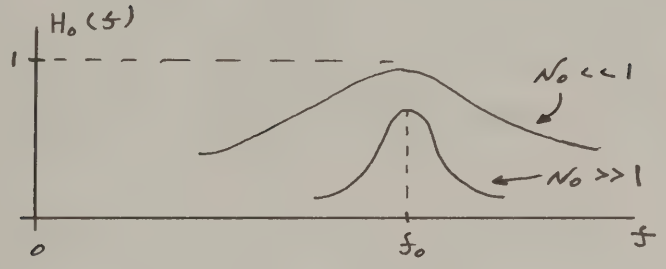
Once again $S_{nn}(f) = \frac{N_0}{2}$ and $S_{sx}(f) = S_{ss}(f)$. Therefore

$$H_o(f) = \frac{S_{sx}(f)}{S_{ss}(f)} = \frac{\frac{A\tau_0}{1 + [2\pi(f - f_0)\tau_0]^2} + \frac{A\tau_0}{1 + [2\pi(f + f_0)\tau_0]^2}}{\frac{A\tau_0}{1 + [2\pi(f - f_0)\tau_0]^2} + \frac{A\tau_0}{1 + [2\pi(f + f_0)\tau_0]^2} + \frac{N_0}{2}}$$

The plot is drawn for N_0 large and N_0 small.

(8.28)

A sketch (single-sided) is



To find e_{min} evaluate

$$e_{min} = \frac{N_o}{2} \int_{-\infty}^{\infty} H_o(f)\, df$$

(9.1)

$I = \log_2\left(\frac{1}{0.3}\right) = 1.737$ bits

$I = \log_e\left(\frac{1}{0.3}\right) = 1.204$ nats

$I = \log_{10}\left(\frac{1}{0.3}\right) = 0.523$ Hartleys.

(9.2)

Let E denote engineer and C denote cafeteria.

From problem statement

$$P(E) = 0.25 \quad , \quad P(C) = 0.60 \quad , \quad P(C/E) = 0.75 .$$

From Bayes' rule

$$P(E/C) = \frac{P(C/E)\,P(E)}{P(C)} = \frac{(0.75)(0.25)}{0.60} = 0.3125$$

$$I(E/C) = \log_2\left(\frac{1}{0.3125}\right) = 1.678 \text{ bits.}$$

(9.3)

$$H(X) = -0.3 \log 0.3 - 0.2 \log 0.2 - 0.2 \log 0.2 - 0.1 \log 0.1 - 0.15 \log 0.15 - 0.05 \log 0.05$$

Using the table of base 2 logs (Appendix C) gives

$$H(X) = 0.521090 + 2(0.464386) + 0.332193 + 0.410545 + 0.216096$$

$$= 2.408696 \text{ bits.}$$

(9.4)

(a) For a single card

$$I(x) = \log_2 52 = 5.70 \text{ bits.}$$

(9.4) (Cont.)

(b) For a pair of cards there are two possibilities. If the first card is replaced before drawing the second card

$$p(x_1, x_2) = p(x_1)\,p(x_2) = \left(\frac{1}{52}\right)^2$$

yielding

$$I(x_1, x_2) = \log_2 (52)^2 = 11.40 \text{ bits.}$$

If the first card is not replaced

$$P(x_1, x_2) = p(x_2/x_1)\,P(x_1) = \frac{1}{52} \cdot \frac{1}{51}$$

and

$$I(x_1, x_2) = \log_2 (52)(51) = 11.37 \text{ bits.}$$

(9.5)

Assume that the channel has n inputs and m outputs. Denote the probability of the j^{th} output by $P(y_j)$. Then

$$P(y_j) = \sum_{i=1}^{N} P(y_j/x_i)\,P(x_i).$$

If the inputs are equally likely

$$P(y_j) = \frac{1}{N}\left[\sum_{i=1}^{N} P(y_j/x_i)\right]$$

The term in brackets is the sum of the elements of the j^{th} column of the channel matrix. Thus, equally likely outputs for equally likely inputs implies that all columns of the channel matrix sum to same value. As a special case, if $m = n$ all columns sum to unity.

(9.6)

Capacity can be determined by maximizing

$$I(X;Y) = H(Y) - H(Y/X)$$

where $H(Y/X)$ can be written

$$H(Y/X) = -\sum_{i=1}^{N} P(x_i) \sum_{j=1}^{M} P(y_j/x_i) \log_2 P(y_j/x_i).$$

If the channel inputs and outputs are statistically independent $P(y_j/x_i) = P(y_j)$ so that

$$H(Y/X) = \sum_{i=1}^{N} P(x_i) \left[-\sum_{j=1}^{M} P(y_j) \log_2 P(y_j) \right]$$

$$= \sum_{i=1}^{N} P(x_i) H(Y) = H(Y) \sum_{i=1}^{N} P(x_i) = H(Y).$$

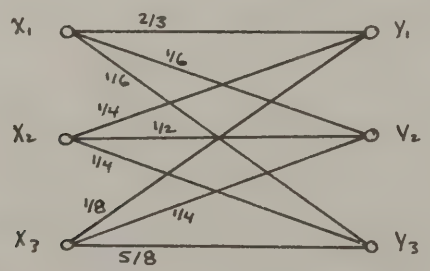
Thus

$$I(X;Y) = H(Y) - H(Y) = 0$$

which implies $C = 0$.

(9.7)

(a) The channel diagram is



(b) Assuming equally likely source outputs, the probabilities associated with the channel outputs are given by

(9.7) (Cont.)

$$P(Y) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 1/4 & 5/8 \end{bmatrix} = \begin{bmatrix} \frac{25}{72} & \frac{22}{72} & \frac{25}{72} \end{bmatrix}$$

so that

$$P(y_1) = \frac{25}{72}, \quad P(y_2) = \frac{22}{72}, \quad P(y_3) = \frac{25}{72}$$

(c) The joint probability matrix is

$$[P(X,Y)] = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 1/4 & 5/8 \end{bmatrix} = \begin{bmatrix} 2/9 & 1/18 & 1/18 \\ 1/12 & 1/6 & 1/12 \\ 1/24 & 1/12 & 5/24 \end{bmatrix}$$

(9.8)

An example of a noiseless channel is illustrated in Fig. 9.4. For this channel

$$P(y_j / x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Thus the channel transition matrix is

$$[P(Y/X)] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The joint probability matrix is also a diagonal matrix. The general element is

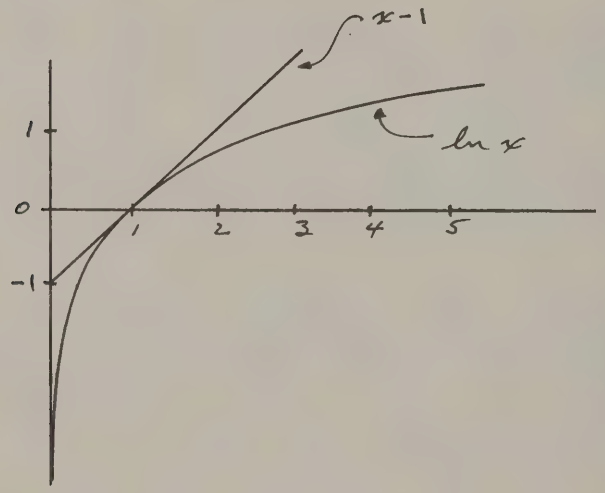
$$P(x_i, y_j) = P(x_i)P(y_j/x_i) = \begin{cases} P(x_i) & i = j \\ 0 & i \neq j \end{cases}$$

Thus it follows that

9.8 (cont.)

$$[p(x,y)] = \begin{bmatrix} p(x_1) & 0 & \cdots & 0 \\ 0 & p(x_2) & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & p(x_n) \end{bmatrix}$$

9.9



Since $I(X;Y) = H(Y) - H(Y|X)$

and

$$-I(X;Y) \leq 0$$

we have

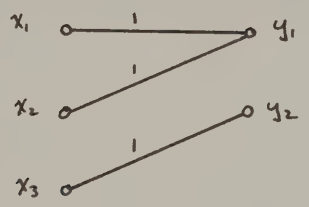
$$H(Y) - H(Y|X) \geq 0$$

or

$$H(Y) \geq H(Y|X)$$

(9.10)

The channel is shown below.



For this channel each input uniquely determines an output. Thus $p(y_j/x_i)$ is either 0 or 1, which results in $H(Y/X) = 0$ (Note that given the input the uncertainty of the output is zero.) Therefore capacity is determined by maximizing

$$I(X;Y) = H(Y).$$

Capacity is achieved for $p(y_1) = p(y_2) = 1/2$ and is

$$C = \log_2 2 = 1 \text{ bit/symbol}$$

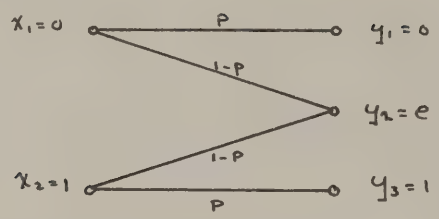
and is achieved for the input probabilities

$$p(x_1) + p(x_2) = 1/2$$

$$p(x_3) = 1/2.$$

(9.11)

The channel is shown below.



Let $p(x_1) = \alpha$ and $p(x_2) = \beta$. Then

$$p(y_1) = \alpha p$$

$$p(y_2) = \alpha(1-p) + \beta(1-p) = (\alpha+\beta)(1-p) = 1-p$$

$$p(y_3) = \beta p.$$

Since $p(x_i/y_j) = p(x_i)\, p(y_j/x_i) / p(y_j)$ we compute

$$p(x_1/y_1) = \frac{\alpha p}{\alpha p} = 1 \qquad p(x_2/y_1) = 0$$

$$p(x_1/y_2) = \frac{\alpha(1-p)}{(1-p)} = \alpha \qquad p(x_2/y_2) = \frac{\beta(1-p)}{(1-p)} = \beta$$

$$p(x_1/y_3) = 0 \qquad p(x_2/y_3) = \frac{\beta p}{\beta p} = 1$$

We shall determine capacity by maximizing

$$I(X;Y) = H(X) - H(X/Y)$$

where

$$H(X/Y) = -\sum_{i=1}^{2}\sum_{j=1}^{3} p(x_i, y_j) \log_2 p(x_i/y_j)$$

and

$$H(X) = -\sum_{i=1}^{2} p(x_i) \log_2 p(x_i).$$

Using the probabilities computed above gives

$$H(X/Y) = -\alpha p \log_2 1 - \alpha(1-p)\log_2\alpha - 0$$

$$- 0 - \beta(1-p)\log_2\beta - \beta p \log_2 1$$

$$= -(1-p)\alpha \log_2\alpha - (1-p)\beta \log_2\beta$$

(9.11) (Cont.)

Grouping terms gives

$$H(X/Y) = (1-p)\left[-\alpha \log_2 \alpha - \beta \log_2 \beta\right].$$

Since

$$H(X) = -\alpha \log_2 \alpha - \beta \log_2 \beta$$

we have

$$H(X/Y) = (1-p)\, H(X)$$

and

$$I(X;Y) = H(X) - (1-p) H(X) = p\, H(X).$$

Capacity is obtained by maximizing $H(X)$. The maximum value of $H(X)$ is one and is achieved for equally likely inputs. Thus

$$C = p \text{ bits/symbol.}$$

This is reasonable. For $p=1$ we have a noiseless channel with $n=2$ yielding

$$C = \log_2 2 = 1 \text{ bit/symbol.}$$

With $p=0$ all inputs are erased and so $C=0$.

(9.12)

The capacity of the channel with the transition matrix

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

(9.12) (Cont.)

is easily computed by maximizing

$$I(X;Y) = H(Y) - H(Y/X).$$

The entropy $H(Y/X)$ can be written

$$H(Y/X) = -\sum_i \sum_j P(x_i)\, P(y_j/x_i) \log_2 P(y_j/x_i)$$

or

$$H(Y/X) = \sum_i P(x_i)\, H(Y/x_i)$$

where

$$H(Y/x_i) = -\sum_j P(y_j/x_i) \log_2 P(y_j/x_i).$$

For the channel given we have, for each x_i

$$H(Y/x_i) = -p\log_2 p - q\log_2 q = K.$$

Thus $H(Y/x_i) = K$; a constant independent of i since each row of the channel matrix has the same probabilities, but not necessarily in the same order. For such a channel

$$H(Y/X) = \sum_i P(x_i)\, K = K \sum_i P(x_i) = K$$

so that

$$I(X;Y) = H(Y) - K.$$

Thus, capacity is achieved when each channel output is equally probable.

We shall now show that for the channel given, equally likely channel outputs are obtained if the channel inputs are equally likely.

(9.12) (cont.)

First,

$$P(y_j) = \sum_i P(x_i)\, P(y_j/x_i)$$
$$= 1/4 \sum_i P(y_j/x_i) = 1/4$$

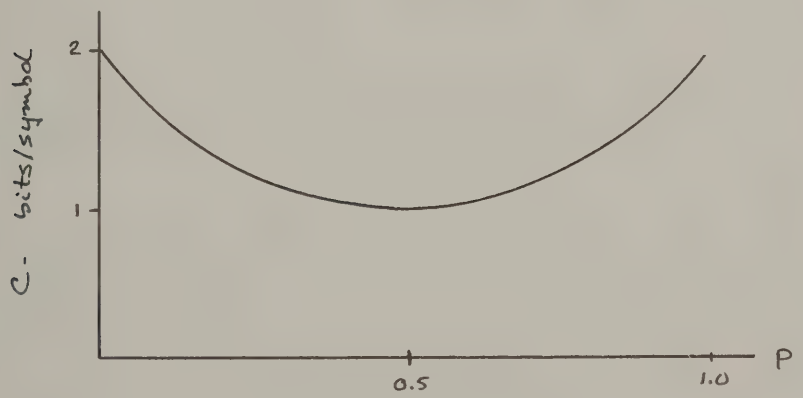
because each column of the channel matrix, for the channel given, sums to unity since q must equal 1-P. Therefore $H(Y) = 2$ and

$$C = 2 - p\log_2 p - q\log_2 q$$

or

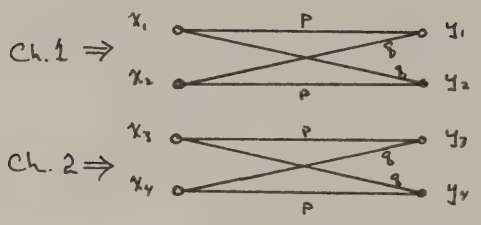
$$C = 2 - H(p) .$$

The capacity is plotted below for various values of p.



This channel is modeled as a pair of binary symmetric channels as shown below. If p = 1 or 0 each input uniquely determines an output. For this case the channel is a noiseless with n=4,

(9.12) (Cont.)



yielding a capacity of 2 bits/symbol. If P = 0.5 the two subchannels (Ch. 1 and Ch. 2) both have zero capacity. However, x_1 and x_2 can be viewed as a single input and x_3 and x_4 can be viewed as a single input. (x_1 and x_2 are used as 0 input and x_3 and x_4 are used as 1 input.) This use of the channel results in a noiseless channel with $n=2$ and results in a capacity of 1 bit/symbol.

Note:

This channel is an excellent example of a general symmetric channel. The capacity of such channels is easily computed. See Gallager (1968) pp. 91-94 for an excellent discussion.

(9.13)

From (9.15)

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(x_i, y_j)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(x_i/y_j)P(y_j)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\left[\log_2 P(x_i/y_j) + \log_2 P(y_j)\right]$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(x_i/y_j) - \sum_{j=1}^{m} P(y_j)\log_2 P(y_j)$$

Applying (9.16) and (9.13) yields (9.17)

$$H(X,Y) = H(X/Y) + H(Y).$$

Equation (9.15) can also be written

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(y_j/x_i)\,P(x_i)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\left[\log_2 P(y_j/x_i) + \log_2 P(x_i)\right]$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(y_j/x_i) - \sum_{i=1}^{n} P(x_i)\log_2 P(x_i)$$

and using (9.14) and (9.12) yields (9.18)

$$H(X,Y) = H(Y/X) + H(X)$$

(9.14)

(a) First we determine $P(m_i)$, $i = 1, 2, 3, 4$.

$$P(m_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^{\sigma} e^{-x^2/2\sigma^2}\,dx = \frac{1}{\sqrt{\pi}}\int_0^{1/\sqrt{2}} e^{-t^2}\,dt$$

$$= \frac{1}{2}\,\mathrm{erf}(0.707) = 0.3413.$$

Also,

$$P(m_3) = 0.5 - 0.3413 = 0.1587.$$

(9.14) (Cont.)

By symmetry $P(m_0) = P(m_3)$ and $P(m_1) = P(m_2)$. Thus the entropy is

$$H(X) = -2[0.3413 \log_2 0.3413 + 0.1587 \log_2 0.1587]$$
$$= 1.902 \text{ bits/symbol.}$$

(b) Since 100 symbols/sec are produced, the information rate is

$$R = 100\, H(X) = 190.2 \text{ bits/sec.}$$

(9.15)

Assumed speech rate = 300 words/min = 5 word/sec.
character rate = (5)(6) = 30 characters/sec.
Thus, the assumed information rate is

$$R = 30\, H(X) \text{ bits/sec,}$$

where $H(X)$ is the entropy of the language in bits/character. We shall assume English language with 27 characters of interest (26 letters and a space to denote word division). Assuming all characters statistically independent and equally likely (not a given assumption) we have

$$R = 30 \log_2 27 = 30(4.755)$$
$$= 142.65 \text{ bits/sec.}$$

The assumptions used in this problem all tend to make the value $R = 142.65$ too

(9.15) (Cont.)

large. A normal speech rate is somewhat less than 300 words/minute (try it). Also, the assumption of statistically independent equally likely characters is clearly not valid. There have been several publications devoted to the structure of language and a short summary is contained in the text by Abramson (1963). It is pointed out that the probability of a space is 0.1859, which implies an average word length of 5.4 characters. Also, a better estimate of $H(X)$ is 3.32 bits/character. (This assumes that the characters are first-order Markov, i.e., the probability of each character is a function of the preceding character) Using these values and a reduced speech rate of 175 words/min yields

$$R = \left(\frac{175}{60}\right)(5.4)\,H(X) = (15.72)(3.32)$$

$$= 52.29 \text{ bits/sec.}$$

(9.16)

Picture Area = $(10^{-2})^2 = 10^{-4}\ m^2$

Area of picture element = $(10^{-4})^2 = 10^{-8}\ m^2$

Number of Elements per picture
$= 10^{-4}/10^{-8} = 10^4$

Ave. Information per picture element
= log 8 = 3 bits (assuming that each gray level is equally likely. This is not a usually valid assumption)

Thus, the average information per picture is $3(10^4) = 30{,}000$ bits.

(9.17)

Assuming 6 characters per word, 1000 words contains 6000 characters. The information in 1000 words is

$$I_W = 6000\ H(X)$$

where $H(X)$ is the entropy of the language in bits per character. Using the results and assumptions of the previous problem, the information in a picture of area A is

$$I_p = \frac{A\ m^2}{10^{-8}\ m^2/\text{pic. ele.}} \cdot 3\ \frac{\text{bits}}{\text{pic. ele.}}$$

(9.17) (cont.)

This yields $I_p = 3A\,(10^8)$ bits

Equating I_p to I_w gives

$$6000\,H(x) = 3A\,(10^8)$$

$$A = 2000\,H(x)\,(10^{-8})$$

$$A = 2H(x)\,(10^{-5})$$

From 9.15, assuming equally likely and statistically independent characters in a word

$$A = 2(10^{-5})\log_2 27$$

$$= 2(4.7549)(10^{-5})$$

$$= 9.51 \times 10^{-5}\ m^2 \approx 10^{-4}\ m^2$$

If from problem 9.15 we assume the value of $H(x)$ to be 2.32 and assume an average wordlength of 5.4 characters, we obtain

$$I_w = (5.4)(1000)\,H(x) = 17,928 \text{ bits}$$

$$I_p = 3A\,(10^8) \text{ bits}$$

$$A = \frac{1.7928\,(10^4)}{3\,(10^8)} = 5.976 \times 10^{-5}\ m^2$$

The assumed size of the picture element has the predominating effect on A. Note also that the addition of color increases the information per picture element and reduces A. For any reasonable set of assumptions 1000 words is a very small picture.

(9.18)

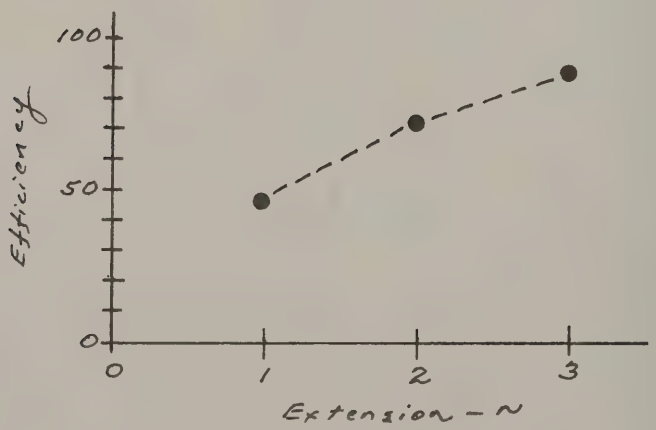
From the example on pages 514-516 the following table is constructed

Code, n	$\bar{L}$	$\bar{L}/n$	Efficiency
1	1	1	46.9%
2	1.29	0.645	72.7%
3	1.598	0.533	88.0%

The last column results since

$$\text{Efficiency} = \frac{\bar{L}_{min}}{\bar{L}} = \frac{nH(x)}{\bar{L}} = \left(\frac{n}{\bar{L}}\right)H(x)$$

where $H(x) = 0.469$ bits/symbol



(9.19)

Source Symbol x_i	$P(x_i)$	Codeword	$P(x_i) \cdot l_i$
x_1	0.125	0 0 0	0.375
x_2	0.125	0 0 1	0.375
x_3	0.125	0 1 0	0.375
x_4	0.125	0 1 1	0.375
x_5	0.125	1 0 0	0.375
x_6	0.125	1 0 1	0.375
x_7	0.125	1 1 0	0.375
x_8	0.125	1 1 1	0.375
			$\bar{L} = 3.000$

The source entropy is $H(X) = 8(\frac{1}{8}) \log_2 8 = 3$ bits/sym. and the efficiency is

$$E = H(X)/(\bar{L} \log_2 D) = 1 = 100\%.$$

(9.20)

Source Symbol x_i	$P(x_i)$	Codeword	$P(x_i) \cdot l_i$
x_1	0.111	0 00	0.333
x_2	0.111	0 0 1	0.333
x_3	0.111	0 1 0	0.333
x_4	0.111	0 1 1	0.333
x_5	0.111	1 00	0.333
x_6	0.111	1 0 1	0.333
x_7	0.111	1 1 0	0.333
x_8	0.111	1 1 1 0	0.444
x_9	0.111	1 1 1 1	0.444

(9.20) (Cont.)

Thus the average word length is

$$\bar{L} = \sum_{i=1}^{N} P(x_i)\, l_i = 3.2219 .$$

The source entropy is

$$H(X) = -9(0.111)\log_2 0.111 = 3.17 \text{ bits/sym.}$$

and the efficiency is

$$E = \frac{H(X)}{\bar{L}\log_2 D} = \frac{3.17}{3.2219} = 0.9839 = 98.39\%.$$

(9.21)

Source Symbol x_i	$P(x_i)$	Codeword	$P(x_i)\cdot l_i$
m_4	0.4	0	0.4
m_3	0.3	10	0.6
m_2	0.2	110	0.6
m_1	0.1	111	0.3
			$\bar{L} = 1.9$

The source entropy is

$$H(X) = -0.4\log_2 0.4 - 0.3\log_2 0.3 - 0.2\log_2 0.2 - 0.1\log_2 0.1$$
$$= 1.8464 \text{ bits/symbol}$$

and the efficiency is

$$E = \frac{H(X)}{\bar{L}\log_2 D} = \frac{1.8464}{1.9} = 0.9718$$
$$= 97.18\%.$$

(9.21) (Cont.)

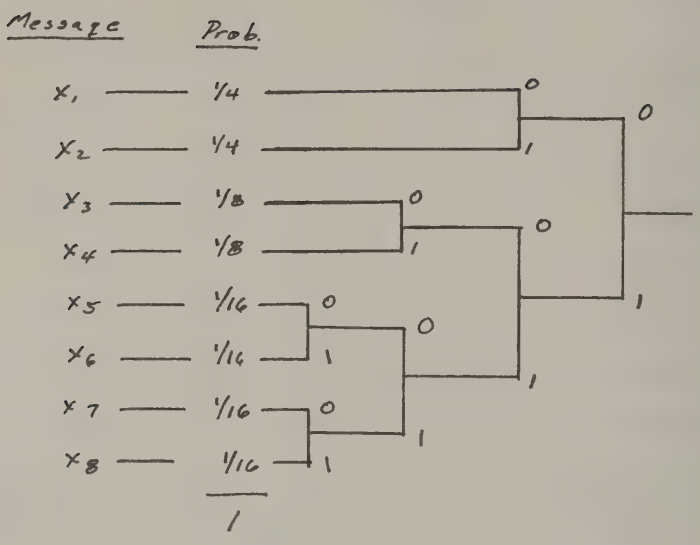
For the second order extension

Source Symbol x_i	$P(x_i)$	Codeword	$P(x_i) \cdot l_i$
$m_4 m_4$	0.16	0 0 0	0.48
$m_4 m_3$	0.12	0 0 1	0.36
$m_3 m_4$	0.12	0 1 0	0.36
$m_3 m_3$	0.09	0 1 1	0.27
$m_4 m_2$	0.08	1 0 0 0	0.32
$m_2 m_4$	0.08	1 0 0 1	0.32
$m_3 m_2$	0.06	1 0 1 0	0.24
$m_2 m_3$	0.06	1 0 1 1	0.24
$m_2 m_2$	0.04	1 1 0 0	0.16
$m_4 m_1$	0.04	1 1 0 1 0	0.20
$m_1 m_4$	0.04	1 1 0 1 1	0.20
$m_3 m_1$	0.03	1 1 1 0 0	0.15
$m_1 m_3$	0.03	1 1 1 0 1	0.15
$m_2 m_1$	0.02	1 1 1 1 0	0.10
$m_1 m_2$	0.02	1 1 1 1 1 0	0.12
$m_1 m_1$	0.01	1 1 1 1 1 1	0.06
			$\bar{L} = 3.73$

From (9.32) the entropy of the second order extension is $2H(X)$, where $H(X) = 1.8464$. Thus, the efficiency is

$$E = 2(1.8464)/3.73 = 0.99 = 99\%$$

9.22



The codewords are

Message	Codeword
x_1	00
x_2	01
x_3	100
x_4	101
x_5	1100
x_6	1101
x_7	1110
x_8	1111

Note that these match Figure 9.10

9.23

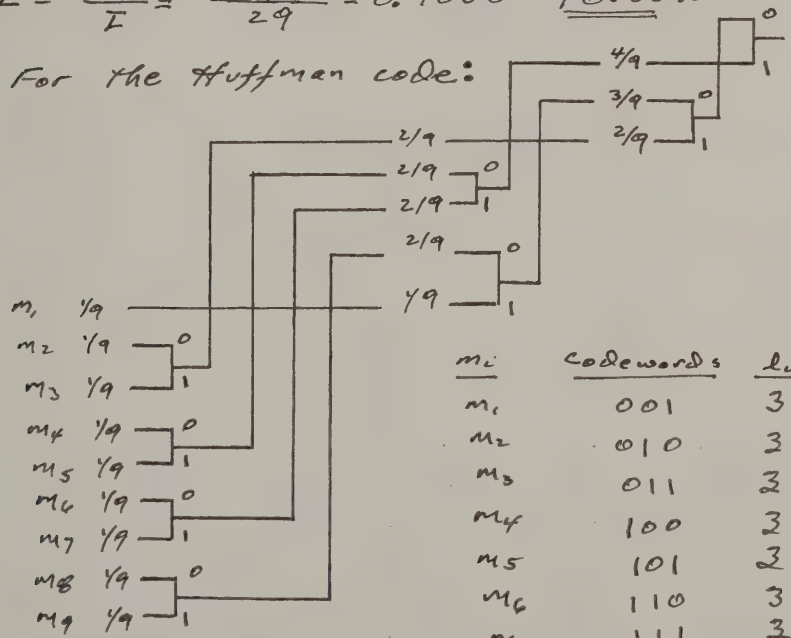
For the Shannon-Fano code:

i	m_i	p_i	Codeword	l_i
1	m_1	1/9	0 0 0	3
2	m_2	1/9	0 0 1	3
3	m_3	1/9	0 1 0	3
4	m_4	1/9	0 1 1	3
5	m_5	1/9	1 0 0	3
6	m_6	1/9	1 0 1	3
7	m_7	1/9	1 1 0 0	4
8	m_8	1/9	1 1 0 1	4
9	m_9	1/9	1 1 1	3

$\bar{L} = \frac{1}{9}(7 \cdot 3 + 2 \cdot 4) = \frac{29}{9}$ $\quad H(X) = \log_2 9$

$E = \frac{H(X)}{\bar{L}} = \frac{9 \log_2 9}{29} = 0.9838 = 98.38\%$

For the Huffman code:

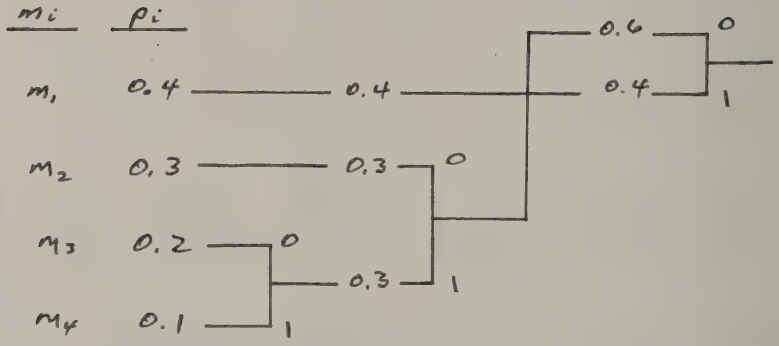


m_i	codewords	l_i
m_1	001	3
m_2	010	3
m_3	011	3
m_4	100	3
m_5	101	3
m_6	110	3
m_7	111	3
m_8	0000	4
m_9	0001	4

Note that the average wordlength of the two codes are equal. Thus they have the same efficiency.

(9.24)

From Problem 9.21 we have



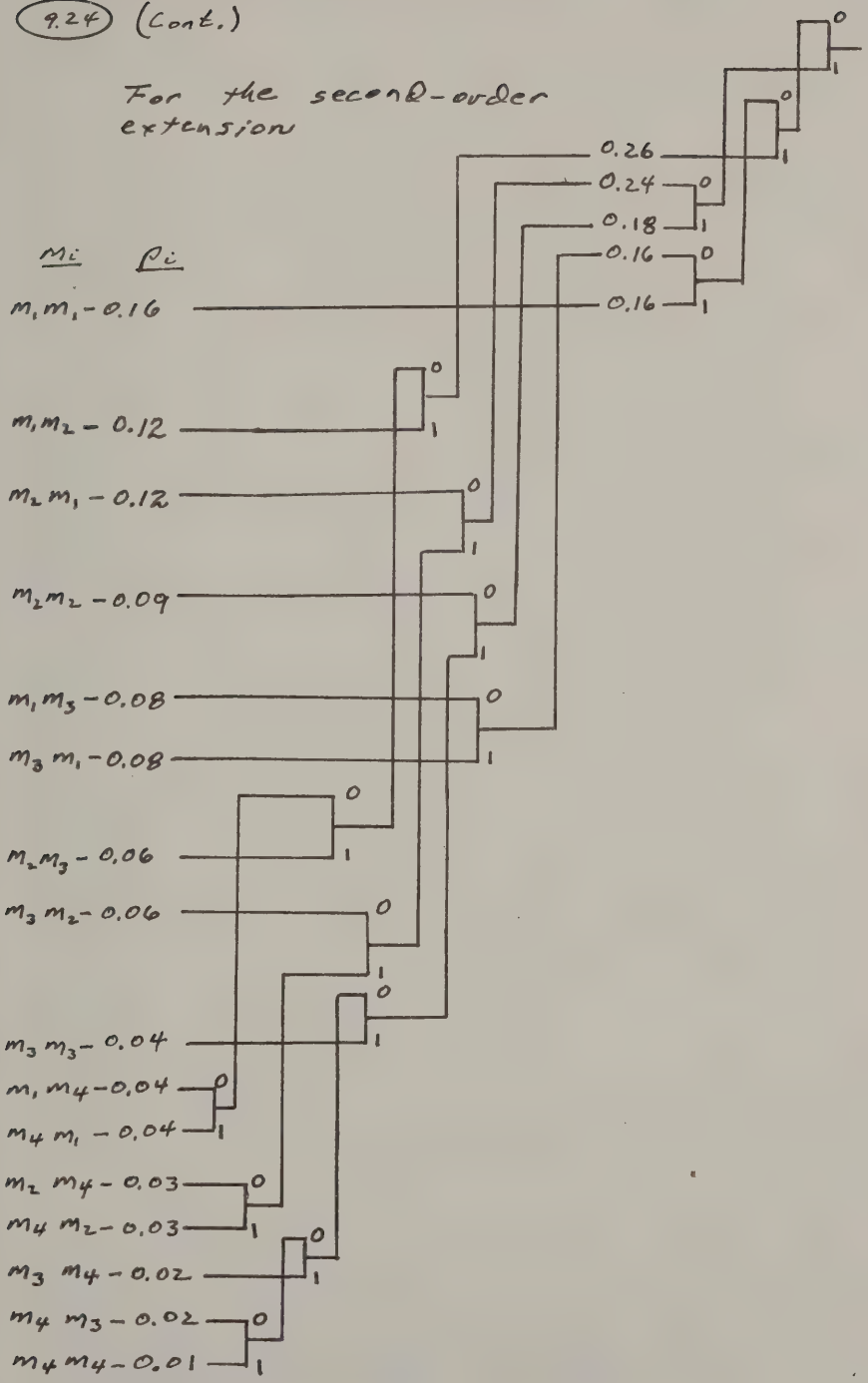
The codewords are

m_i	Codeword
m_1	1
m_2	0 0
m_3	0 1 0
m_4	0 1 1

Note that the average word length is $\bar{L} = 1.9$. The entropy is of course the same so that $H(x) = 1.8464$ as found in Problem 9.24. Thus the efficiency is 97.18% as before. Note that the codewords are different but could be made the same by assigning the symbols at the transitions differently.

9.24 (Cont.)

For the second-order extension



(9.24) (cont.)

The result is

Message	Prob	Codeword	l_i
$m_1 m_1$	0.16	001	3
$m_1 m_2$	0.12	011	3
$m_2 m_1$	0.12	100	3
$m_2 m_2$	0.09	110	3
$m_1 m_3$	0.08	0000	4
$m_3 m_1$	0.08	0001	4
$m_2 m_3$	0.06	0101	4
$m_3 m_2$	0.06	1010	4
$m_3 m_3$	0.04	1111	4
$m_1 m_4$	0.04	01000	5
$m_4 m_1$	0.04	01001	5
$m_2 m_4$	0.03	10110	5
$m_4 m_2$	0.03	10111	5
$m_3 m_4$	0.02	11101	5
$m_4 m_3$	0.02	111000	6
$m_4 m_4$	0.01	111001	6

$$\bar{L} = \sum_{i=1}^{16} p_i l_i = 3.73$$

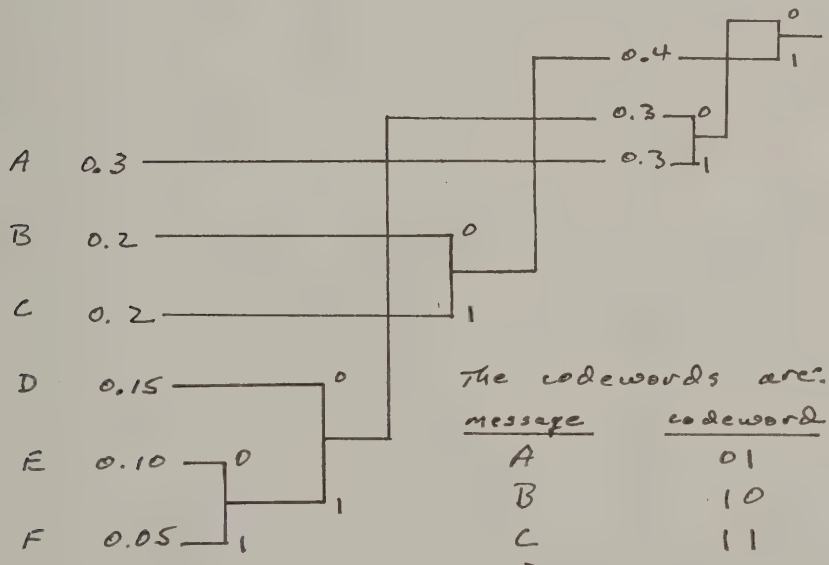
The average wordlength is the same as that obtained in Problem 9.21. The entropy was computed in 9.21 and found to be 2(1.8464). Thus $E = 99\%$. The results of 9.21 and 9.24 are identical.

(9.25) For the Shannon-Fano code -

message	prob.	codeword	l_i
A	0.3	0 0	2
B	0.2	0 1	2
C	0.2	1 0	2
D	0.15	1 1 0	3
E	0.10	1 1 1 0	4
F	0.05	1 1 1 1	4

$$\bar{L} = \sum p_i l_i = 2(0.3) + 2(2)(0.2) + 3(0.15) + 4(0.10) + 4(0.05)$$
$$= 2.45 \text{ sym/codeword}$$

For the Huffman code -



The codewords are:

message	codeword
A	0 1
B	1 0
C	1 1
D	0 0 0
E	0 0 1 0
F	0 0 1 1

Since the set of wordlengths, l_i, are equal to the set of wordlengths for the Shannon-Fano code, $\bar{L} = 2.45$ for the Huffman code also.

(9.26)

For the Shannon-Fano code

message	Prob.	Codeword	l_i
m_1	1/11	0 0 0	3
m_2	1/11	0 0 1	3
m_3	1/11	0 1 0	3
m_4	1/11	0 1 1 0	4
m_5	1/11	0 1 1 1	4
m_6	1/11	1 0 0	3
m_7	1/11	1 0 1 0	4
m_8	1/11	1 0 1 1	4
m_9	1/11	1 1 0	3
m_{10}	1/11	1 1 1 0	4
m_{11}	1/11	1 1 1 1	4

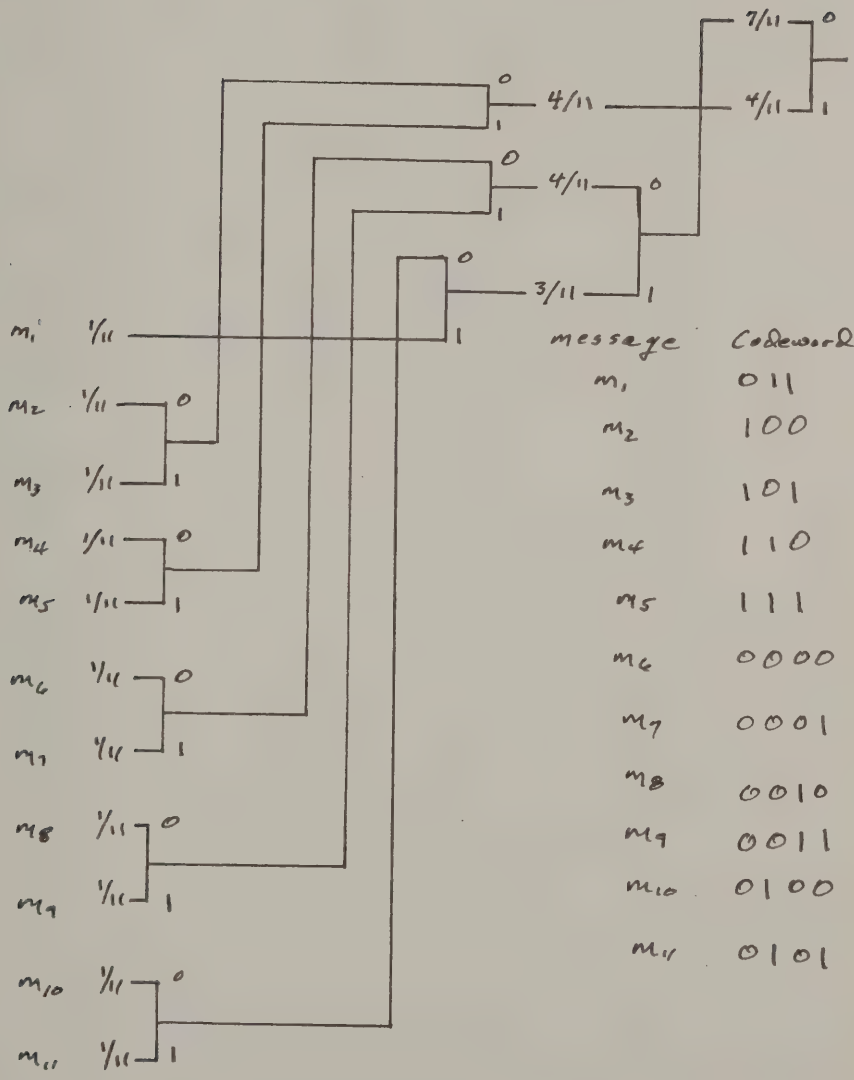
The average wordlength is

$$\bar{L} = \frac{1}{11}(3+3+3+4+4+3+4+4+3+4+4)$$

$$= \frac{1}{11}(39) = \frac{39}{11} = 3.545 \text{ symbols/codeword}$$

9.26 (Cont.)

For the Huffman code-



message	Codeword
m_1	011
m_2	100
m_3	101
m_4	110
m_5	111
m_6	0000
m_7	0001
m_8	0010
m_9	0011
m_{10}	0100
m_{11}	0101

As with the Shannon-Fano code 5 codewords have length 3 and 6 codewords have length 4. Thus $\bar{L} = 3.545$ for both codes

(9.27)

Source Symbol x_i	$P(x_i)$	Codeword	$P(x_i) \cdot l_i$
$m_1 m_1$	0.16	00	0.32
$m_1 m_2$	0.08	01	0.16
$m_2 m_1$	0.08	02	0.16
$m_1 m_3$	0.08	10	0.16
$m_3 m_1$	0.08	110	0.24
$m_1 m_4$	0.04	111	0.12
$m_4 m_1$	0.04	120	0.12
$m_1 m_5$	0.04	121	0.12
$m_5 m_1$	0.04	122	0.12
$m_2 m_2$	0.04	200	0.12
$m_2 m_3$	0.04	201	0.12
$m_3 m_2$	0.04	202	0.12
$m_3 m_3$	0.04	210	0.12
$m_2 m_4$	0.02	2110	0.08
$m_4 m_2$	0.02	2111	0.08
$m_2 m_5$	0.02	2120	0.08
$m_5 m_2$	0.02	2121	0.08
$m_3 m_4$	0.02	2200	0.08
$m_4 m_3$	0.02	2201	0.08
$m_3 m_5$	0.02	2210	0.08
$m_5 m_3$	0.02	2211	0.08
$m_4 m_4$	0.01	2220	0.04
$m_4 m_5$	0.01	2221	0.04
$m_5 m_4$	0.01	22220	0.05
$m_5 m_5$	0.01	22221	0.05

(9.27) (Cont.)

The average wordlength is

$$\bar{L} = \sum_{i=1}^{N} P(x_i) \cdot l_i = 2.82 .$$

The entropy is

$$H(X) = -0.16 \log_2 0.16 - 4(0.08) \log_2 0.08 - 8(0.04) \log_2 0.04 - 8(0.02) \log_2 0.02 - 4(0.01) \log_2 0.01$$

$$= 4.244 \text{ bits / symbol}$$

and the efficiency is

$$E = \frac{H(X)}{\bar{L} \log_2 D} = \frac{4.244}{2.82 \log_2 3} = 0.9495 = 94.95\% .$$

(9.28)

From (9.33)

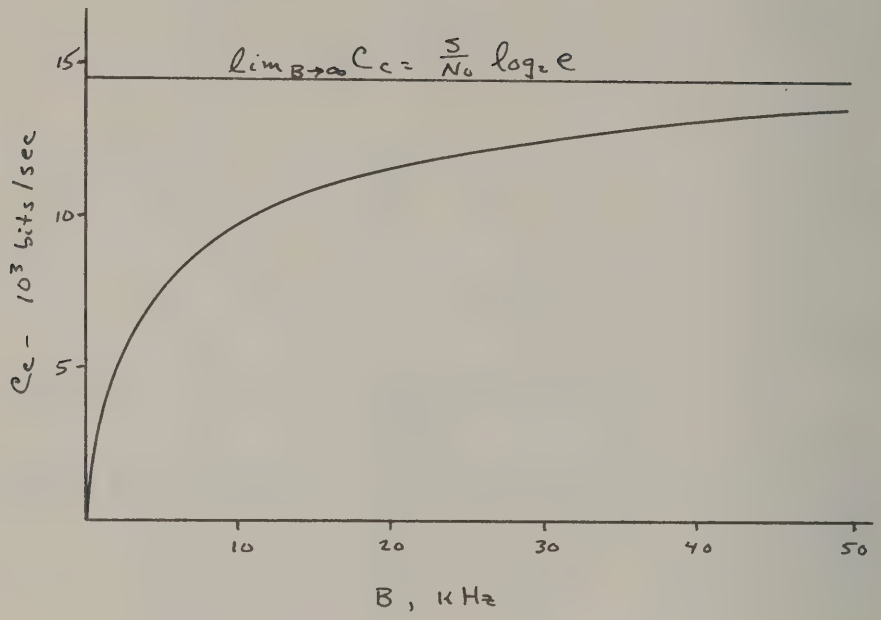
$$C_c = B \log_2 \left(1 + \frac{S}{N_0 B}\right) = B \log_2 \left(1 + \frac{10^4}{B}\right)$$

The following table of values is obtained

B	C_c
1000 Hz	3459 bits/second
5000	7924
10,000	10,000
50,000	13,152
100,000	13,750
∞	$\frac{S}{N_0} \log_2 e = 14,426$

The plot of these values follows.

(9.28) (Cont.)


$\lim_{B\to\infty} C_c = \frac{S}{N_0}\log_2 e$
15
10
5
C_c - 10^3 bits/sec
10
20
30
40
50
B, KHz

9.29

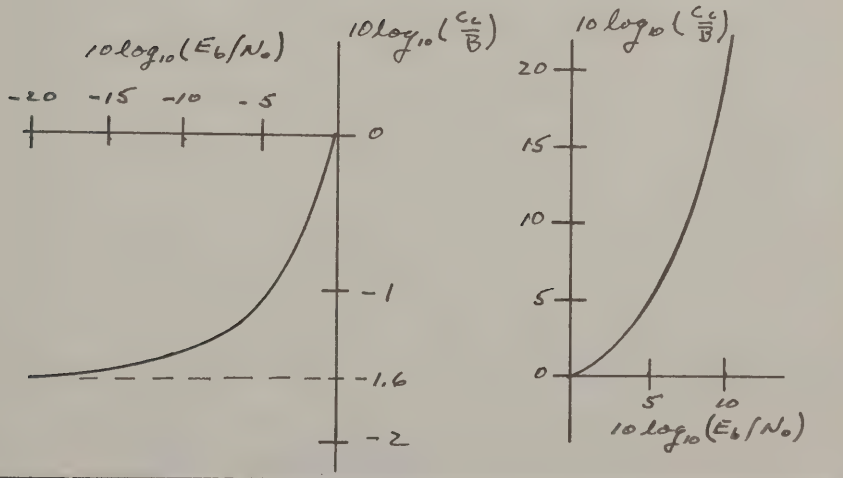
$$C_c = B \log_2\left(1 + \frac{S}{N_0 B}\right) = B \log_2\left(1 + \frac{E_b}{N_0}\frac{C_c}{B}\right)$$

$$\frac{C_c}{B} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C_c}{B}\right)$$

$$2^{C_c/B} = 1 + \left(\frac{E_b}{N_0}\right)\left(\frac{C_c}{B}\right)$$

$$\left(\frac{E_b}{N_0}\right) = \left(\frac{B}{C_c}\right)\left[2^{(C_c/B)} - 1\right]$$

C_c/B	$10\log_{10} C_c/B$	E_b/N_0	$10\log_{10} E_b/N_0$
0.01	-20	0.70	-1.58
0.05	-13	0.71	-1.52
0.1	-10	0.72	-1.44
0.2	-6.99	0.74	-1.29
0.5	-3.01	0.83	-0.82
0.75	-1.25	0.91	-0.41
1	0	1	0
2	3.01	1.5	1.76
4	6.02	3.75	5.74
7	8.45	18.1	12.59
10	10	102.3	20.10

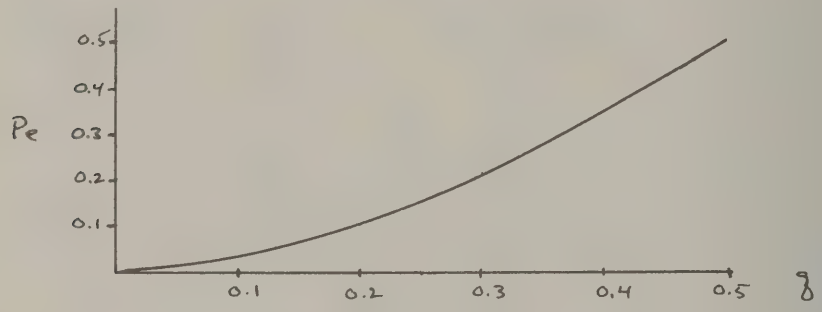


(9.30)

From (9.44) the probability of error for a rate $1/3$ repetition code is

$$P_e = 3(1-p)^2 p + (1-p)^3 = 3q^2 p + q^3$$

where $(1-p)$ is the probability of error on a single transmission and $q = 1-p$.



(9.31)

For a rate 1/5 repetition code the probability of error, P_e, is equal to the probability of 3, 4, or 5 errors in 5 transmissions. Let the probability of error on a single transmission be $q = (1-p)$. Then

$$P_e = \binom{5}{3} p^2 q^3 + \binom{5}{4} p q^4 + \binom{5}{5} q^5$$
$$= 10p^2q^3 + 5pq^4 + q^5.$$

(9.32)

We know that a Hamming code allows correction of all single errors (all columns of the parity check matrix are distinct and there is no all zero column). Thus

$$1 = \tfrac{1}{2}(d_m - 1) \Rightarrow d_m = 3$$

It is an interesting (but unnecessary) exercise to compute the distance between all combinations of two codewords in the set of $2^K = 16$ codewords generated by a (7, 4) Hamming code. The parity check matrix is

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Assuming a codeword of the form $x_1, x_2, x_3, x_4, x_5, x_6, x_7$, the parity check equations are

$$x_1 = x_3 \oplus x_5 \oplus x_7$$

(9.32) (Cont.)

$$x_2 = x_3 \oplus x_6 \oplus x_7$$

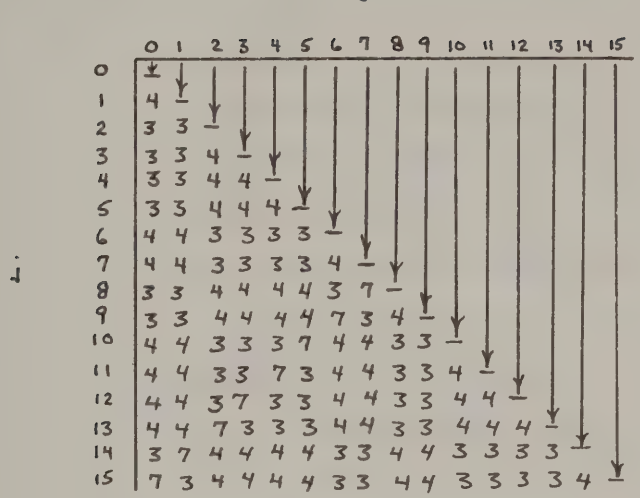
$$x_4 = x_5 \oplus x_6 \oplus x_7 .$$

This yields the 16 codewords given below.

Codeword	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	0	0	0	0	0
1	1	1	0	1	0	0	1
2	0	1	0	1	0	1	0
3	1	0	0	0	0	1	1
4	1	0	0	1	1	0	0
5	0	1	0	0	1	0	1
6	1	1	0	0	1	1	0
7	0	0	0	1	1	1	1
8	1	1	1	0	0	0	0
9	0	0	1	1	0	0	1
10	1	0	1	1	0	1	0
11	0	1	1	0	0	1	1
12	0	1	1	1	1	0	0
13	1	0	1	0	1	0	1
14	0	0	1	0	1	1	0
15	1	1	1	1	1	1	1

The distance between codewords i and j, denoted d_{ij}, is given in the following table. Note that since $d_{ij} = d_{ji}$, only half of the table need be filled.

(9.32) (Cont.)



As can be seen from the table above, the minimum distance d_{ij} for $i \neq j$ is 3. Thus, a (7, 4) Hamming code is a distance 3 code.

(9.33)

The parity check matrix for a (15, 11) Hamming code is constructed by recognizing that the i^{th} column is the binary representation of the number i, $1 \leq i \leq 15$. Thus,

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The general codeword is written

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15}$.

(9.33) (Cont.)

Within this codeword the parity checks are x_1, x_2, x_4, and x_8. The parity equations are

$$x_1 = x_3 \oplus x_5 \oplus x_7 \oplus x_9 \oplus x_{11} \oplus x_{13} \oplus x_{15}$$

$$x_2 = x_3 \oplus x_6 \oplus x_7 \oplus x_{10} \oplus x_{11} \oplus x_{14} \oplus x_{15}$$

$$x_4 = x_5 \oplus x_6 \oplus x_7 \oplus x_{12} \oplus x_{13} \oplus x_{14} \oplus x_{15}$$

$$x_8 = x_9 \oplus x_{10} \oplus x_{11} \oplus x_{12} \oplus x_{13} \oplus x_{14} \oplus x_{15}.$$

Note that each parity check is the sum of an odd number (7) of information symbols. Thus if each information symbol is 1, we have

$$x_1 = x_2 = x_4 = x_8 = 1$$

so that an all ones input yields an all ones codeword. With an error in the 7th position the syndrome is computed as

$$[S] = [H][1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since the syndrome is the binary representation of 7, a single error in the 7th position is indicated.

(9.34)

The parity check matrix is

$$[H] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

We see that the parity checks are in the third, sixth, and seventh positions. Thus for

(9.34) (Cont.)

the general codeword $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

the parity equations are

$$x_3 = x_1 \oplus x_4 \oplus x_5$$

$$x_6 = x_1 \oplus x_2 \oplus x_5$$

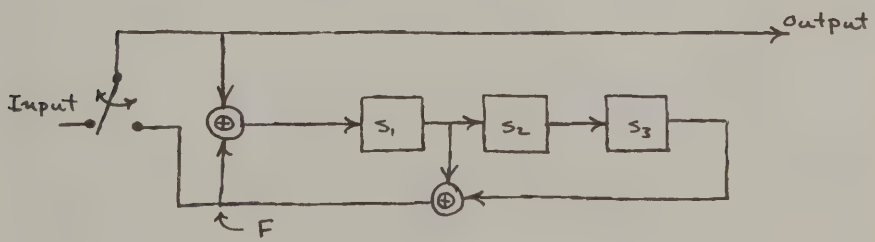
$$x_7 = x_1 \oplus x_2 \oplus x_4.$$

This yields the codewords

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	0	0	0	0
0	0	1	0	1	1	0
0	0	1	1	0	0	1
0	0	0	1	1	1	1
0	1	0	0	0	1	1
0	1	1	0	1	0	1
0	1	1	1	0	1	0
0	1	0	1	1	0	0
1	0	1	0	0	1	1
1	0	0	0	1	0	1
1	0	0	1	0	1	0
1	0	1	1	1	0	0
1	1	1	0	0	0	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1
1	1	1	1	1	1	1

(9.35)

The encoder is shown below.



Assume that the codeword is of the form

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$

(9.35) (Cont.)

First the parity check equations must be determined. The output for arbitrary input is found as shown in table below.

Shift	S_1	S_2	S_3	F	Output
1	X_1	0	0	X_1	X_1
2	$X_1 \oplus X_2$	X_1	0	$X_1 \oplus X_2$	X_2
3	$X_1 \oplus X_2 \oplus X_3$	$X_1 \oplus X_2$	X_1	$X_2 \oplus X_3$	X_3
4	$X_2 \oplus X_3 \oplus X_4$	$X_1 \oplus X_2 \oplus X_3$	$X_1 \oplus X_2$	$X_1 \oplus X_3 \oplus X_4$	X_4
5	0	$X_2 \oplus X_3 \oplus X_4$	$X_1 \oplus X_2 \oplus X_3$	$X_1 \oplus X_2 \oplus X_3$	$X_1 \oplus X_3 \oplus X_4$
6	0	0	$X_2 \oplus X_3 \oplus X_4$	$X_2 \oplus X_3 \oplus X_4$	$X_1 \oplus X_2 \oplus X_3$
7	0	0	0	0	$X_2 \oplus X_3 \oplus X_4$

Thus we have the parity check equations

$$X_5 = X_1 \oplus X_3 \oplus X_4$$

$$X_6 = X_1 \oplus X_2 \oplus X_3$$

$$X_7 = X_2 \oplus X_3 \oplus X_4$$

This yields the following $2^K = 16$ codewords.

X_1	X_2	X_3	X_4	X_5	X_6	X_7
0	0	0	0	0	0	0
1	0	0	0	1	1	0
1	1	0	0	1	0	1
1	1	1	0	0	1	0
1	1	1	1	1	1	1
0	0	0	1	1	0	1
0	0	1	1	0	1	0
0	1	1	1	0	0	1

X_1	X_2	X_3	X_4	X_5	X_6	X_7
1	0	1	1	1	0	0
1	1	0	1	0	0	0
1	0	0	1	0	1	1
0	1	1	0	1	0	0
0	1	0	0	0	1	1
0	0	1	0	1	1	1
0	1	0	1	1	1	0
1	0	1	0	0	0	1

(9.35) (Cont.)

The general codeword is of the form

$$X = (x_1, x_2, x_3, x_4, x_1 \oplus x_3 \oplus x_4, x_1 \oplus x_2 \oplus x_3, x_2 \oplus x_3 \oplus x_4).$$

A cyclic shift gives the sequence

$$X_s = (x_2 \oplus x_3 \oplus x_4, x_1, x_2, x_3, x_4, x_1 \oplus x_3 \oplus x_4, x_1 \oplus x_2 \oplus x_3)$$

or $Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$ where y_i is given by

$$y_1 = x_2 \oplus x_3 \oplus x_4 \qquad y_5 = x_4$$

$$y_2 = x_1 \qquad y_6 = x_1 \oplus x_3 \oplus x_4$$

$$y_3 = x_2 \qquad y_7 = x_1 \oplus x_2 \oplus x_3$$

$$y_4 = x_3$$

We must now check to see if y_5, y_6, and y_7 obey the parity equations.

$$y_5 = y_1 \oplus y_3 \oplus y_4 = (x_2 \oplus x_3 \oplus x_4) \oplus x_2 \oplus x_3 = x_4 = y_5$$

$$y_6 = y_1 \oplus y_2 \oplus y_3 = (x_2 \oplus x_3 \oplus x_4) \oplus x_1 \oplus x_2 = x_1 \oplus x_3 \oplus x_4 = y_6$$

$$y_7 = y_2 \oplus y_3 \oplus y_4 = x_1 \oplus x_2 \oplus x_3 = y_7.$$

The sixteen codewords contain <u>all</u> possible combinations of 0 and 1 in the first four positions. Thus X and Y are general and we have shown that a cyclic permutation of a codeword yields another codeword. Note that the parity check matrix for the encoder is (from the parity equations)

$$[H] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(9.35) (Cont.)

Thus we see that this code is essentially a Hamming code. The position of the parity checks have been changed. Since the columns of [H] are distinct, the encoder generates a distance three code which allows correction of all single errors. The same is true of the encoder in Fig. 9.13.

(9.36)

For a (15,11) Hamming code the symbol error probability for the uncoded case is given by (9.64) with K=11. Thus,

$$q_u = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{S\,T_w}{11\,N_0}}\right)$$

so that the uncoded word error probability is

$$P_{eu} = 1 - (1-q_u)^{11} = \sum_{i=1}^{11} \binom{11}{i} q_u^i (1-q_u)^{11-i}.$$

For the coded case, the symbol error probability becomes (9.65) with $n=15$ so that

$$q_c = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{S\,T_w}{15\,N_0}}\right).$$

The word error probability is

$$P_{ec} = \sum_{i=2}^{15} \binom{15}{i} q_c^i (1-q_c)^{15-i}.$$

In Part III at the end of this solutions manual, a computer program

(9.36) (Cont.)

is given for evaluating P_{eu} and P_{ec}. The tabular results are given on the following two pages. This is followed by the required performance curves.

Also shown is the approximation to P_{ec} assuming that all word errors result from two symbol errors and that terms of the form $(1-q_c)^{15-i}$ are negligible. This approximation is

$$P_{ec} \approx \binom{15}{2} q_c^2 = 105\, q_c^2.$$

The approximation is clearly poor for small $5T_w/N_0$.

(9.36) (Cont.)

********** N= 15 K= 11 E= 1 **********

****************** UNCODED CASE ******************

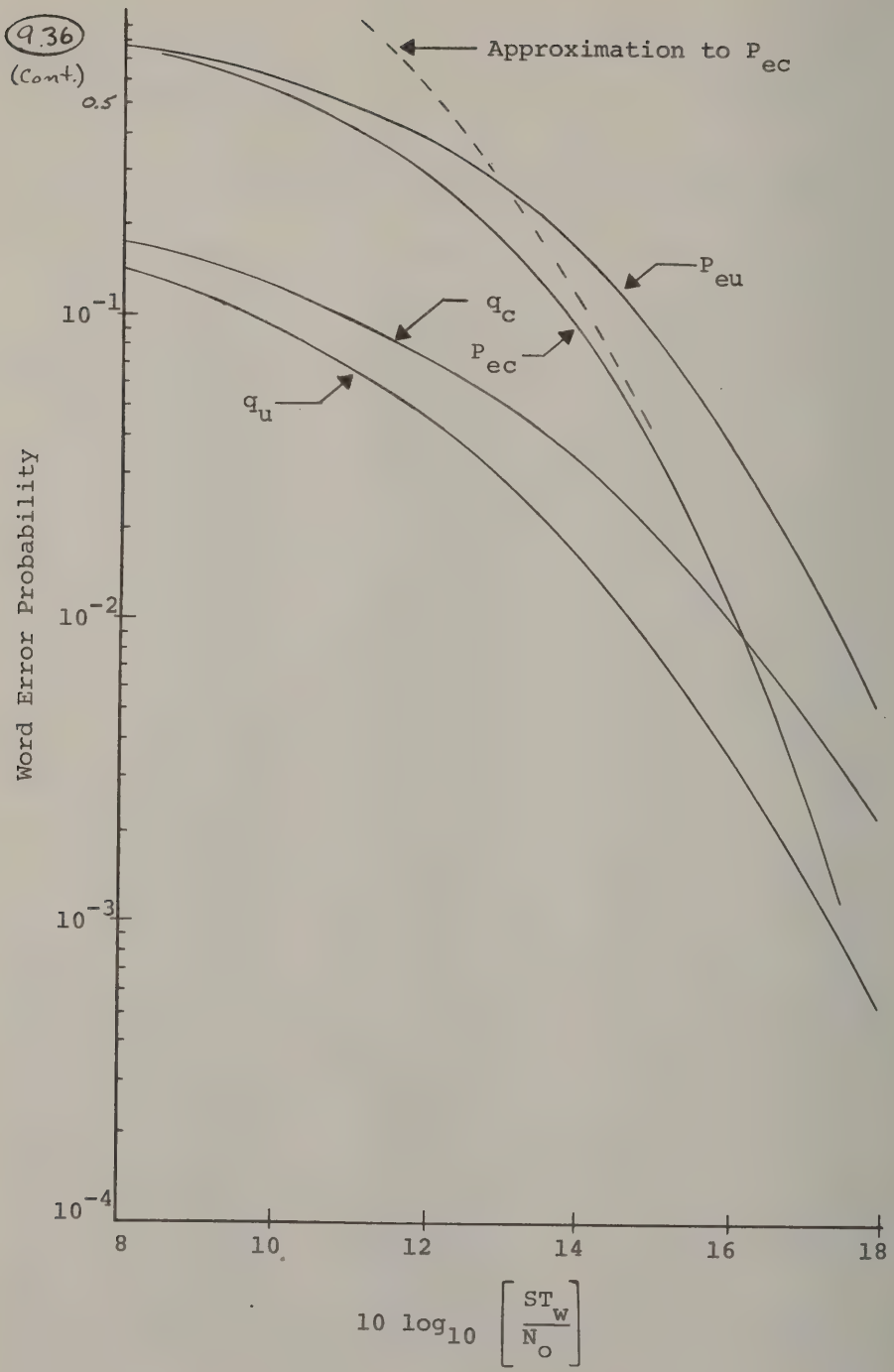
10LOG(STW/NO)	OU	PEU
0.0	0.3349D+00	0.9887D+00
0.5000D+00	0.3258D+00	0.9869D+00
0.1000D+01	0.3162D+00	0.9847D+00
0.1500D+01	0.3062D+00	0.9821D+00
0.2000D+01	0.2957D+00	0.9788D+00
0.2500D+01	0.2848D+00	0.9750D+00
0.3000D+01	0.2735D+00	0.9702D+00
0.3500D+01	0.2617D+00	0.9645D+00
0.4000D+01	0.2496D+00	0.9575D+00
0.4500D+01	0.2370D+00	0.9490D+00
0.5000D+01	0.2241D+00	0.9387D+00
0.5500D+01	0.2109D+00	0.9262D+00
0.6000D+01	0.1974D+00	0.9110D+00
0.6500D+01	0.1837D+00	0.8928D+00
0.7000D+01	0.1699D+00	0.8710D+00
0.7500D+01	0.1560D+00	0.8451D+00
0.8000D+01	0.1421D+00	0.8147D+00
0.8500D+01	0.1283D+00	0.7791D+00
0.9000D+01	0.1147D+00	0.7383D+00
0.9500D+01	0.1015D+00	0.6919D+00
0.1000D+02	0.8876D-01	0.6403D+00
0.1050D+02	0.7660D-01	0.5838D+00
0.1100D+02	0.6515D-01	0.5234D+00
0.1150D+02	0.5451D-01	0.4602D+00
0.1200D+02	0.4480D-01	0.3960D+00
0.1250D+02	0.3608D-01	0.3325D+00
0.1300D+02	0.2841D-01	0.2717D+00
0.1350D+02	0.2182D-01	0.2155D+00
0.1400D+02	0.1630D-01	0.1653D+00
0.1450D+02	0.1180D-01	0.1224D+00
0.1500D+02	0.8246D-02	0.8706D-01
0.1550D+02	0.5544D-02	0.5932D-01
0.1600D+02	0.3568D-02	0.3856D-01
0.1650D+02	0.2187D-02	0.2380D-01
0.1700D+02	0.1269D-02	0.1387D-01
0.1750D+02	0.6930D-03	0.7596D-02
0.1800D+02	0.3533D-03	0.3879D-02
0.1850D+02	0.1668D-03	0.1833D-02
0.1900D+02	0.7226D-04	0.7946D-03
0.1950D+02	0.2843D-04	0.3127D-03
0.2000D+02	0.1004D-04	0.1104D-03

(9.36) (Cont.)

********** N= 15 K= 11 E= 1 **********

********** CODED CASE **********		
10LOG(STW/NO)	QC	PEC
0.0	0.3575D+00	0.9877D+00
0.5000D+00	0.3495D+00	0.9857D+00
0.1000D+01	0.3410D+00	0.9832D+00
0.1500D+01	0.3322D+00	0.9802D+00
0.2000D+01	0.3229D+00	0.9765D+00
0.2500D+01	0.3132D+00	0.9720D+00
0.3000D+01	0.3030D+00	0.9665D+00
0.3500D+01	0.2924D+00	0.9598D+00
0.4000D+01	0.2814D+00	0.9516D+00
0.4500D+01	0.2699D+00	0.9416D+00
0.5000D+01	0.2581D+00	0.9293D+00
0.5500D+01	0.2458D+00	0.9144D+00
0.6000D+01	0.2331D+00	0.8963D+00
0.6500D+01	0.2201D+00	0.8744D+00
0.7000D+01	0.2068D+00	0.8481D+00
0.7500D+01	0.1933D+00	0.8167D+00
0.8000D+01	0.1795D+00	0.7799D+00
0.8500D+01	0.1656D+00	0.7370D+00
0.9000D+01	0.1517D+00	0.6879D+00
0.9500D+01	0.1378D+00	0.6326D+00
0.1000D+02	0.1241D+00	0.5718D+00
0.1050D+02	0.1106D+00	0.5063D+00
0.1100D+02	0.9756D-01	0.4378D+00
0.1150D+02	0.8498D-01	0.3684D+00
0.1200D+02	0.7302D-01	0.3004D+00
0.1250D+02	0.6180D-01	0.2364D+00
0.1300D+02	0.5144D-01	0.1787D+00
0.1350D+02	0.4202D-01	0.1292D+00
0.1400D+02	0.3362D-01	0.8885D-01
0.1450D+02	0.2628D-01	0.5781D-01
0.1500D+02	0.2002D-01	0.3540D-01
0.1550D+02	0.1481D-01	0.2027D-01
0.1600D+02	0.1061D-01	0.1079D-01
0.1650D+02	0.7335D-02	0.5301D-02
0.1700D+02	0.4868D-02	0.2386D-02
0.1750D+02	0.3089D-02	0.9752D-03
0.1800D+02	0.1863D-02	0.3586D-03
0.1850D+02	0.1062D-02	0.1173D-03
0.1900D+02	0.5682D-03	0.3373D-04
0.1950D+02	0.2832D-03	0.8400D-05
0.2000D+02	0.1304D-03	0.1782D-05

9.36
(Cont.)



Performance Curves for a (15,11) Hamming Code

(9.37)

From (9.66) the probability of two symbol errors in a 7 bit codeword is

$$P\{2 \text{ errors}\} = \binom{7}{2} q_c^2 (1-q_c)^5 = 21 q_c^2 (1-q_c)^5$$

and the probability of three symbol errors is

$$P\{3 \text{ errors}\} = \binom{7}{3} q_c^3 (1-q_c)^4 = 35 q_c^3 (1-q_c)^4$$

where, from (9.65) with $n = 7$

$$q_c = \tfrac{1}{2}\operatorname{erfc}\left(\sqrt{\tfrac{5}{7}\tfrac{T_w}{N_o}}\right).$$

The ratio of $P\{3 \text{ errors}\}$ to $P\{2 \text{ errors}\}$ is

$$\frac{P(3 \text{ errors})}{P(2 \text{ errors})} = \frac{35 q_c^3 (1-q_c)^4}{21 q_c^2 (1-q_c)^5} = \frac{5}{3}\frac{q_c}{1-q_c}.$$

The values for this ratio are given below for $S T_w/N_o$ between 8dB and 14 dB, the range of interest in Fig. 9.15 and Example 9.12.

$S T_w/N_o$	q_c	$P\{3 \text{ er}\}/P\{2 \text{ er}\}$
8 dB	0.0897	0.164
9	0.0660	0.118
10	0.0455	0.0794
11	0.0290	0.0497
12	0.0167	0.0283
13	0.0085	0.0143
14	0.0037	0.0062

Thus for high signal-to-noise ratios the approximation that $P\{3 \text{ errors}\}$ is negligible compared to $P\{2 \text{ errors}\}$ is certainly valid. For values of $S T_w/N_o < 10$ this approximation can result in considerable error.

(9.38)

The parity check matrix for a (7,4) Hamming code is

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The parity checks are in positions 1, 2 and 4. Moving the parity checks to positions 5, 6 and 7 yields

$$[H] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Note that all colums of [H] are distinct and therefore all single errors will be corrected. The distance 3 property is preserved. The generator matrix is

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(9.39)

The code is clearly systematic. The codeword is of the form $a_1 a_2 a_3 a_4 c_1 c_2 c_3$ The parity equations are

$$c_1 = h_{11} a_1 \oplus h_{12} a_2 \oplus h_{13} a_3 \oplus h_{14} a_4$$
$$c_2 = h_{21} a_1 \oplus h_{22} a_2 \oplus h_{23} a_3 \oplus h_{24} a_4$$
$$c_3 = h_{31} a_1 \oplus h_{32} a_2 \oplus h_{33} a_3 \oplus h_{34} a_4$$

We shall input a_1, a_2, a_3 and a_4 in sequence and determine the output.

(9.39) (Cont.)

Shift	S_1	S_2	S_3	Output
1	a_1	0	0	a_1
2	a_2	a_1	0	a_2
3	$a_3 \oplus a_1$	a_2	a_1	a_3
4	$a_1 \oplus a_2 \oplus a_4$	$a_1 \oplus a_3$	a_2	a_4
5	0	$a_1 \oplus a_2 \oplus a_4$	$a_1 \oplus a_3$	$a_1 \oplus a_2 \oplus a_3$
6	0	0	$a_1 \oplus a_2 \oplus a_4$	$a_2 \oplus a_3 \oplus a_4$
7	0	0	0	$a_1 \oplus a_2 \oplus a_4$

Thus:

$$c_1 = 1 \cdot a_1 \oplus 1 \cdot a_2 \oplus 1 \cdot a_3 \oplus 0 \cdot a_4$$
$$c_2 = 0 \cdot a_1 \oplus 1 \cdot a_2 \oplus 1 \cdot a_3 \oplus 1 \cdot a_4$$
$$c_3 = 1 \cdot a_1 \oplus 1 \cdot a_2 \oplus 0 \cdot a_3 \oplus 1 \cdot a_4$$

and

$$[G] = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&1&1&0 \\ 0&1&1&1 \\ 1&1&0&1 \end{bmatrix} \qquad [H] = \begin{bmatrix} 1&1&1&0&1&0&0 \\ 0&1&1&1&0&1&0 \\ 1&1&0&1&0&0&1 \end{bmatrix}$$

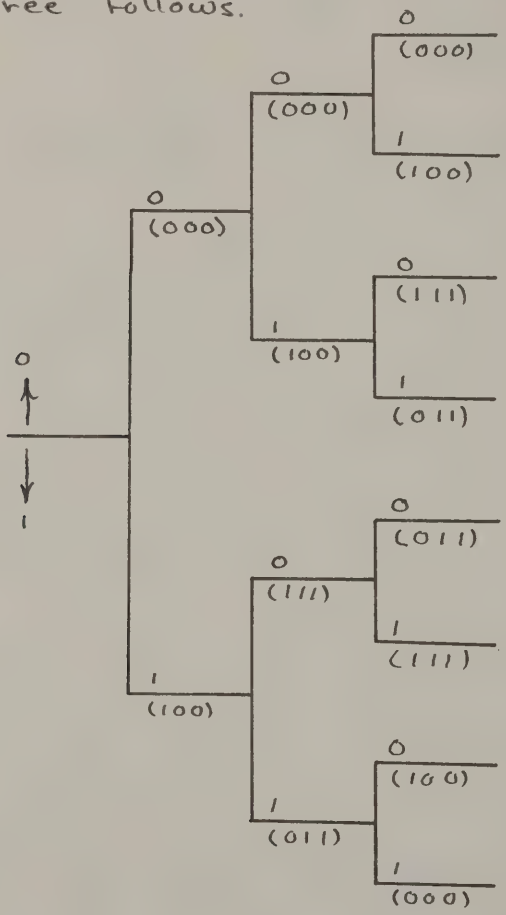
with 1101001 received, the syndrome is

$$[S] = \begin{bmatrix} 1&1&1&0&1&0&0 \\ 0&1&1&1&0&1&0 \\ 1&1&0&1&0&0&1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the zero syndrome denotes no errors. Thus 1101001 is the most likely transmitted sequence. This agrees with Figure 9.14. The student should repeat the problem with 1101011 received.

(9.40)

The output for each shift is $v_1\, v_2\, v_3$ where $v_1 = s_1 \oplus s_2$, $v_2 = s_2 \oplus s_3$, $v_3 = s_2 \oplus s_3$. The codetree follows.



This is not a good code. For decoding, the two segments diverging from a node should lie as far apart in Hamming distance as possible, i.e., distance K where code rate is $\frac{1}{K}$. For this code diverging segments lie distance one apart.

(9.41)

Constraint Span, $K = 4$

Need three states.

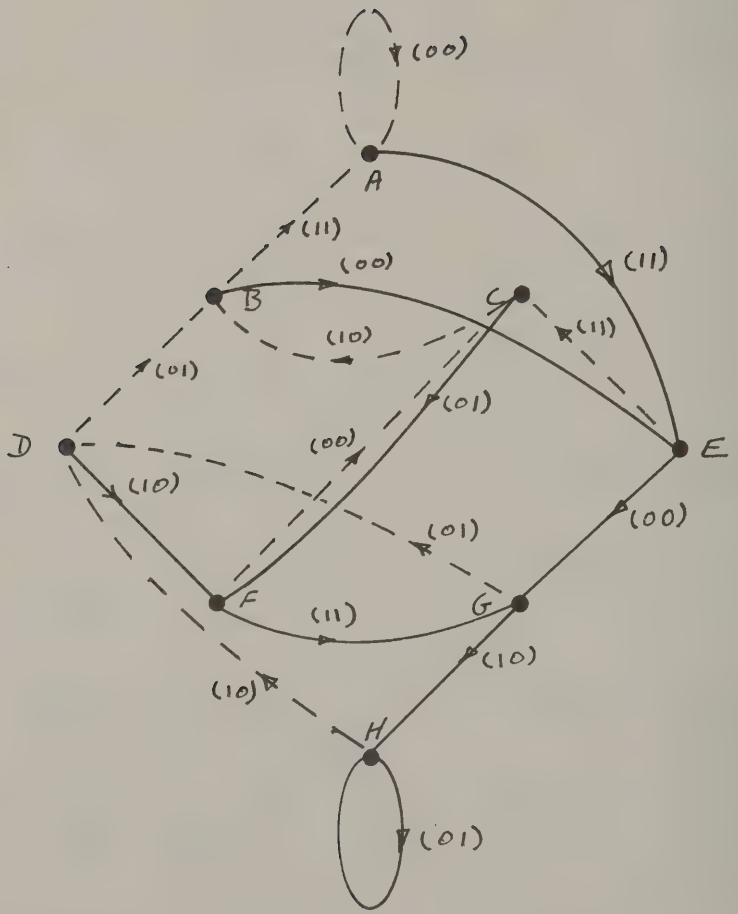
$v_1 = s_1 \oplus s_2 \oplus s_3 \oplus s_4$

$v_2 = s_1 \oplus s_2 \oplus s_4$

Previous state	s'_1	s'_2	s'_3	Input	Current s_1	s_2	s_3	s_4	state	output
A	0	0	0	0	0	0	0	0	A	0 0
				1	1	0	0	0	E	1 1
B	0	0	1	0	0	0	0	1	A	1 1
				1	1	0	0	1	E	0 0
C	0	1	0	0	0	0	1	0	B	1 0
				1	1	0	1	0	F	0 1
D	0	1	1	0	0	0	1	1	B	0 1
				1	1	0	1	1	F	1 0
E	1	0	0	0	0	1	0	0	C	1 1
				1	1	1	0	0	G	0 0
F	1	0	1	0	0	1	0	1	C	0 0
				1	1	1	0	1	G	1 1
G	1	1	0	0	0	1	1	0	D	0 1
				1	1	1	1	0	H	1 0
H	1	1	1	0	0	1	1	1	D	1 0
				1	1	1	1	1	H	0 1

The state diagram is on the following page.

9.41 (Cont.)

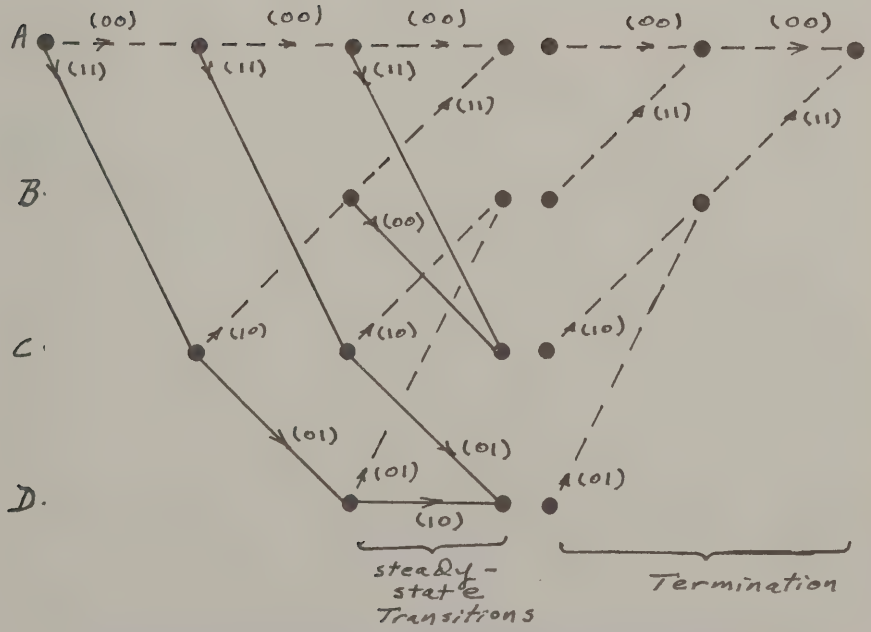


9.42

From the encoder $v_1 = s_1 \oplus s_2 \oplus s_3$

$v_2 = s_1 \oplus s_3$

Previous state	s_1	s_2	Input	Current s_1'	s_2'	s_3'	State	Output v_1	v_2
A	0	0	0	0	0	0	A	0	0
			1	1	0	0	C	1	1
B	0	1	0	0	0	1	A	1	1
			1	1	0	1	C	0	0
C	1	0	0	0	1	0	B	1	0
			1	1	1	0	D	0	1
D	1	1	0	0	1	1	B	0	1
			1	1	1	1	D	1	0



(9.43)

In a practical situation one would use the smallest possible value of n. Thus we shall assume a $(7,4)$ Hamming code and determine the necessary value of ℓ, the number of codewords read into the table. The source produces symbols at the rate of 1000 symbols per second. Since these symbols are encoded using a $(7,4)$ code, the signaling rate through the channel is

$$(1000)\left(\frac{7}{4}\right) = 1750 \text{ symbols/sec.}$$

This symbol rate yields the same information rate into and out of the encoder.

Since the error bursts are 0.01 seconds in duration, a maximum of 18 symbols can be affected by a single burst. Thus, the table must contain at least $\ell = 18$ codewords. In order for the burst to be corrected, the burst must be followed by at least

$$(n-1)\,\ell = (6)(18) = 108$$

error free transmissions. Thus the minimum time between bursts is

$$\frac{108}{1750} = 0.0617 \text{ seconds.}$$

(9.44)

For a (15,7) BCH code the symbol error probability for the uncoded case is (9.64) with K=7. Thus

$$q_u = \tfrac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{5T_w}{7N_0}}\right)$$

so that the uncoded word error probability is

$$P_{eu} = 1-(1-q_u)^7 = \sum_{i=1}^{7}\binom{7}{i} q_u^i (1-q_u)^{7-i}.$$

For the coded case, the symbol error probability is (9.65) with n=15. Thus

$$q_c = \tfrac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{5T_w}{7N_0}}\right).$$

Since both single and double errors can be corrected, the word error probability for the coded case is

$$P_{ec} = \sum_{i=3}^{15}\binom{15}{i} q_c^i (1-q_c)^{15-i}.$$

As in Problem 9.36, the computer program at the end of this manual is used to evaluate P_{eu} and P_{ec}. The tabular results are given on the following two pages. The required curves follow. Also shown on the performance curves is the approximation to P_{ec} assuming that all word errors result from three symbol errors, and that terms of the form $(1-q_c)^{15-i}$ are negligible. The approximation is $P_{ec} \approx \binom{15}{3} q_c^3 = 455\, q_c^3$. This approximation is once again clearly useless for small $5T_w/N_0$.

(9.44) (Cont.)

********** N= 15 K= 7 E= 2 **********

****************** UNCODED CASE ******************

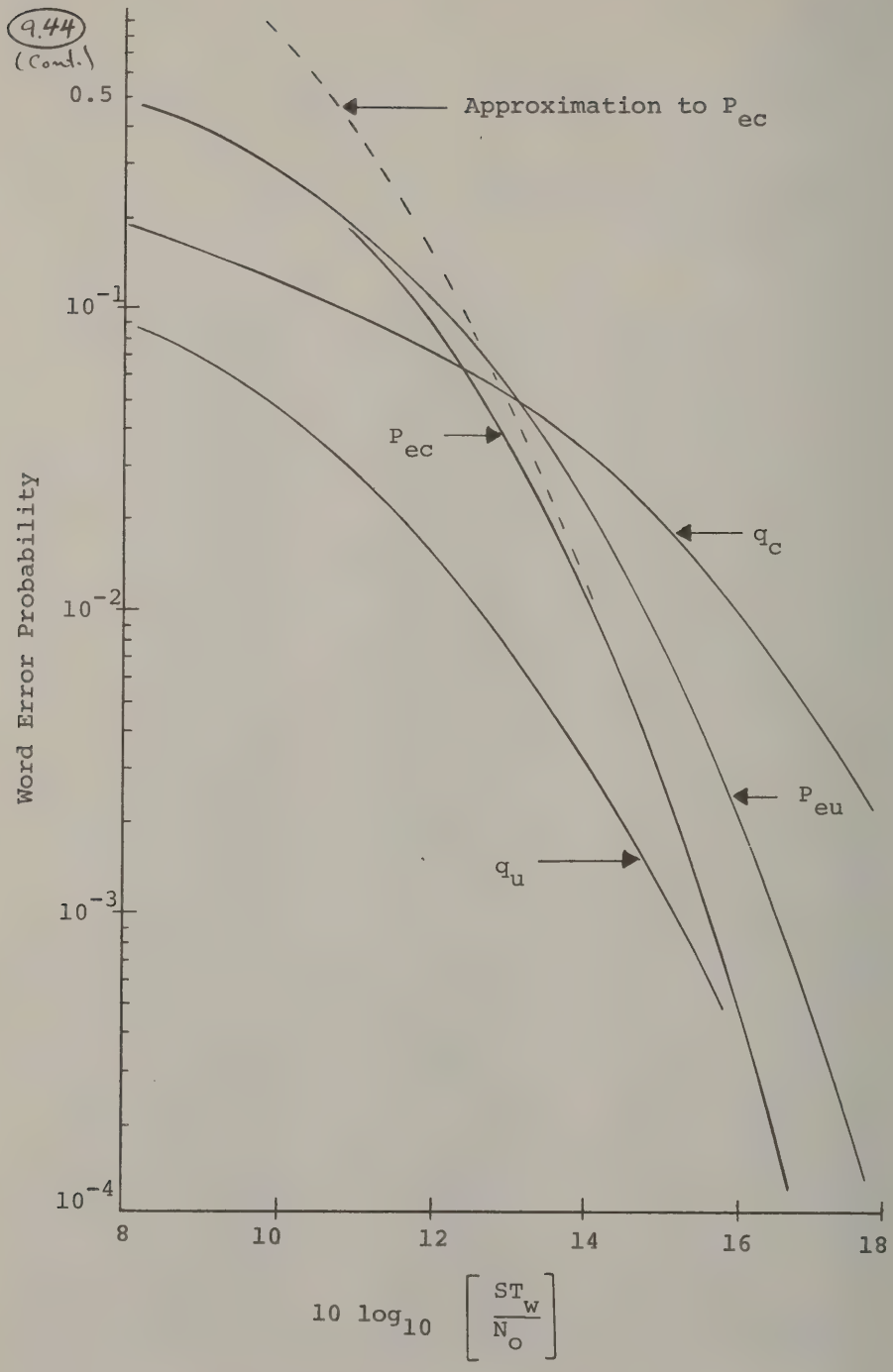
10LOG(STW/NO)	QU	PEU
0.0	0.2965D+00	0.9147D+00
0.5000D+00	0.2856D+00	0.9051D+00
0.1000D+01	0.2743D+00	0.8940D+00
0.1500D+01	0.2626D+00	0.8815D+00
0.2000D+01	0.2505D+00	0.8671D+00
0.2500D+01	0.2380D+00	0.8508D+00
0.3000D+01	0.2251D+00	0.8323D+00
0.3500D+01	0.2119D+00	0.8112D+00
0.4000D+01	0.1985D+00	0.7874D+00
0.4500D+01	0.1848D+00	0.7607D+00
0.5000D+01	0.1709D+00	0.7307D+00
0.5500D+01	0.1570D+00	0.6975D+00
0.6000D+01	0.1431D+00	0.6607D+00
0.6500D+01	0.1293D+00	0.6206D+00
0.7000D+01	0.1157D+00	0.5772D+00
0.7500D+01	0.1025D+00	0.5308D+00
0.8000D+01	0.8969D-01	0.4820D+00
0.8500D+01	0.7748D-01	0.4314D+00
0.9000D+01	0.6597D-01	0.3798D+00
0.9500D+01	0.5527D-01	0.3283D+00
0.1000D+02	0.4548D-01	0.2781D+00
0.1050D+02	0.3669D-01	0.2302D+00
0.1100D+02	0.2894D-01	0.1858D+00
0.1150D+02	0.2227D-01	0.1459D+00
0.1200D+02	0.1667D-01	0.1110D+00
0.1250D+02	0.1210D-01	0.8166D-01
0.1300D+02	0.8479D-02	0.5786D-01
0.1350D+02	0.5718D-02	0.3934D-01
0.1400D+02	0.3693D-02	0.2556D-01
0.1450D+02	0.2272D-02	0.1580D-01
0.1500D+02	0.1324D-02	0.9233D-02
0.1550D+02	0.7264D-03	0.5074D-02
0.1600D+02	0.3723D-03	0.2603D-02
0.1650D+02	0.1768D-03	0.1237D-02
0.1700D+02	0.7712D-04	0.5397D-03
0.1750D+02	0.3057D-04	0.2140D-03
0.1800D+02	0.1089D-04	0.7621D-04
0.1850D+02	0.3439D-05	0.2407D-04
0.1900D+02	0.9493D-06	0.6645D-05
0.1950D+02	0.2253D-06	0.1577D-05
0.2000D+02	0.4515D-07	0.3161D-06

(9.44) (cont.)

********** N= 15 K= 7 E= 2 **********

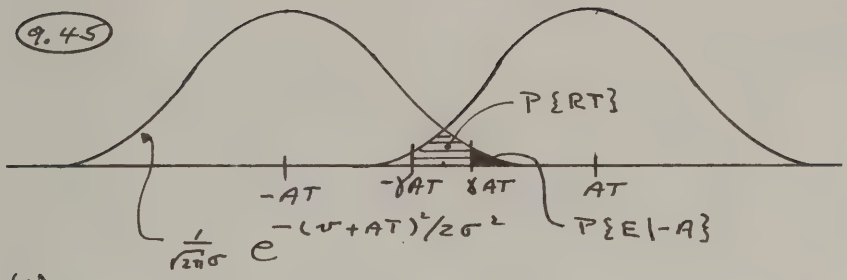
10LOG(STW/NO)	QC	PEC
0.0	0.3575D+00	0.9451D+00
0.5000D+00	0.3495D+00	0.9377D+00
0.1000D+01	0.3410D+00	0.9292D+00
0.1500D+01	0.3322D+00	0.9193D+00
0.2000D+01	0.3229D+00	0.9076D+00
0.2500D+01	0.3132D+00	0.8940D+00
0.3000D+01	0.3030D+00	0.8782D+00
0.3500D+01	0.2924D+00	0.8598D+00
0.4000D+01	0.2814D+00	0.8383D+00
0.4500D+01	0.2699D+00	0.8135D+00
0.5000D+01	0.2581D+00	0.7850D+00
0.5500D+01	0.2458D+00	0.7523D+00
0.6000D+01	0.2331D+00	0.7152D+00
0.6500D+01	0.2201D+00	0.6735D+00
0.7000D+01	0.2068D+00	0.6272D+00
0.7500D+01	0.1933D+00	0.5763D+00
0.8000D+01	0.1795D+00	0.5215D+00
0.8500D+01	0.1656D+00	0.4634D+00
0.9000D+01	0.1517D+00	0.4032D+00
0.9500D+01	0.1378D+00	0.3425D+00
0.1000D+02	0.1241D+00	0.2829D+00
0.1050D+02	0.1106D+00	0.2264D+00
0.1100D+02	0.9756D-01	0.1747D+00
0.1150D+02	0.8498D-01	0.1294D+00
0.1200D+02	0.7302D-01	0.9151D-01
0.1250D+02	0.6180D-01	0.6143D-01
0.1300D+02	0.5144D-01	0.3891D-01
0.1350D+02	0.4202D-01	0.2310D-01
0.1400D+02	0.3362D-01	0.1276D-01
0.1450D+02	0.2628D-01	0.6515D-02
0.1500D+02	0.2002D-01	0.3047D-02
0.1550D+02	0.1481D-01	0.1294D-02
0.1600D+02	0.1061D-01	0.4943D-03
0.1650D+02	0.7335D-02	0.1681D-03
0.1700D+02	0.4868D-02	0.5024D-04
0.1750D+02	0.3089D-02	0.1304D-04
0.1800D+02	0.1863D-02	0.2893D-05
0.1850D+02	0.1062D-02	0.5397D-06
0.1900D+02	0.5682D-03	0.8303D-07
0.1950D+02	0.2832D-03	0.1031D-07
0.2000D+02	0.1304D-03	0.1007D-08

9.44 (Cont.)



Performance Curves for a (15,7) BCH Code

(9.45)



(a)

In the above diagram RT denotes retransmission. The variance σ^2 is the variance and is given by (7.5). Thus

$$\sigma^2 = \frac{1}{2} N_0 T$$

the probability of retransmission, conditioned on -A transmitted is

$$P\{RT|-A\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\gamma AT}^{\gamma AT} e^{-(v+AT)^2/2\sigma^2} \, dv$$

with the change of variable $u = (v+AT)/\sqrt{2}\sigma$

$$P\{RT|-A\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{AT(1-\gamma)/\sqrt{2}\sigma}^{AT(1+\gamma)/\sqrt{2}\sigma} e^{-u^2} \sqrt{2}\sigma \, du$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{AT(1+\gamma)/\sqrt{2}\sigma} e^{-u^2} du - \frac{1}{\sqrt{\pi}} \int_0^{AT(1-\gamma)/\sqrt{2}\sigma} e^{-u^2} du$$

$$= \frac{1}{2} erf\left[AT(1+\gamma)/\sqrt{2}\sigma\right] - \frac{1}{2} erf\left[AT(1-\gamma)/\sqrt{2}\sigma\right]$$

Since $\frac{AT}{\sqrt{2}\sigma} = \frac{AT}{\sqrt{2}}\sqrt{\frac{2}{N_0 T}} = \frac{\sqrt{2}}{\sqrt{2}}\sqrt{\frac{A^2T}{N_0}} = \sqrt{z}$

where

$$z = \frac{A^2 T}{N_0}$$

Thus

$$P\{RT|-A\} = \frac{1}{2} erf\left(\sqrt{z}(1+\gamma)\right) - \frac{1}{2} erf\left(\sqrt{z}(1-\gamma)\right)$$

(9.45) (Cont.)

By symmetry $P\{RT|-A\} = P\{RT|+A\}$. This yields

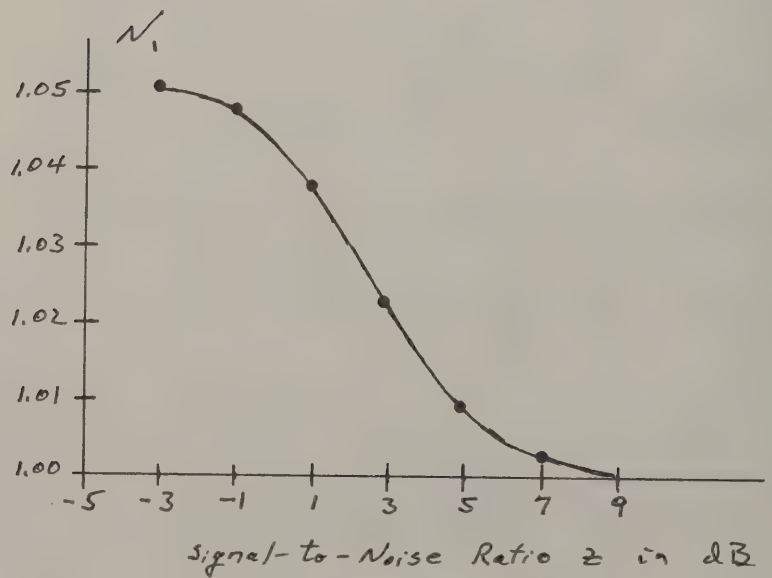
$$P_z = P\{RT\} = \tfrac{1}{2}\,erf\left[\sqrt{z}(1+\gamma)\right] - \tfrac{1}{2}\,erf\left[\sqrt{z}(1-\gamma)\right]$$

with $\gamma = 0.1$ we have

$$P_z = \tfrac{1}{2}\,erf\left[1.1\sqrt{z}\right] - \tfrac{1}{2}\,erf\left[0.9\sqrt{z}\right]$$

We can construct the following table

z dB	P_z	$N_1 = \frac{1}{1-P_z}$
-3	4.839×10^{-2}	1.051
-1	4.552×10^{-2}	1.048
1	3.618×10^{-2}	1.038
3	2.210×10^{-2}	1.023
5	8.972×10^{-3}	1.009
7	1.942×10^{-3}	1.002
9	1.613×10^{-3}	1.000



(9.45) (cont)

(b)

The overall probability of error is $P_E = \frac{P_1}{1-P_2}$ where P_1 is computed as follows.

$$P(E|-A) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\gamma AT}^{\infty} e^{-(v+AT)^2/2\sigma^2}\, dv$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{AT(1+\gamma)/2\sigma}^{\infty} e^{-u^2} \sqrt{2}\sigma\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_{\sqrt{z}(1+\gamma)}^{\infty} e^{-u^2}\, du$$

$$= \tfrac{1}{2}\operatorname{erfc}\left[\sqrt{z}(1+\gamma)\right] = \tfrac{1}{2}\operatorname{erfc}\left[1.1\sqrt{z}\right]$$

Without feedback the error probability of error is P_{EN}

$$P_{EN} = \tfrac{1}{2}\operatorname{erfc}\left(\sqrt{z}\right)$$

We can construct the following table

z dB	N_1	P_E (with feedback)	P_{EN} (without feedback)
−3	1.051	1.423×10^{-1}	1.584×10^{-1}
−1	1.048	8.675×10^{-2}	1.038×10^{-1}
1	1.038	4.179×10^{-2}	5.628×10^{-2}
3	1.023	1.43×10^{-2}	2.288×10^{-2}
5	1.009	2.860×10^{-3}	5.954×10^{-3}
7	1.002	2.488×10^{-4}	7.727×10^{-4}
9	1.000	5.822×10^{-6}	3.364×10^{-5}

The decrease in error probability is obvious with a very moderate increase in N_1 the required number of transmissions.

(9.46)

From the previous problem (9.45) the probability of retransmission is

$$P_2 = \frac{1}{2}\,erfc\left[\sqrt{z}\,(1+\gamma)\right] + \frac{1}{2}\,erfc\left[\sqrt{z}\,(1-\gamma)\right]$$

The overall probability of error is

$$P_E = \frac{P_1}{1 - P_2}$$

where

$$P_1 = \frac{1}{2}\,erfc\left[\sqrt{z}\,(1+\gamma)\right]$$

The probability of error without feedback is

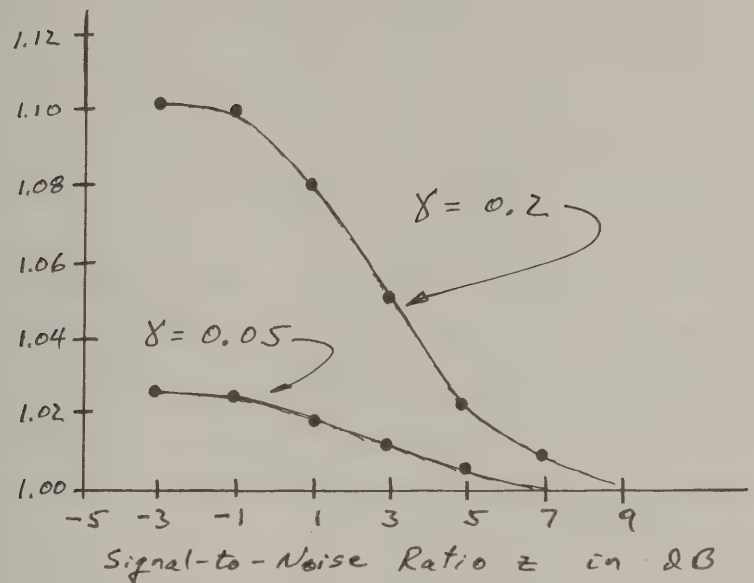
$$P_{EN} = \frac{1}{2}\,erfc\left(\sqrt{z}\right)$$

(a) We form the following table

	$\gamma = 0.05$		$\gamma = 0.2$	
z dB	P_2	$N_1 = \frac{1}{1-P_2}$	P_2	$N_1 = \frac{1}{1-P_2}$
−3	2.420×10^{-1}	1.025	9.679×10^{-2}	1.107
−1	2.273×10^{-2}	1.023	9.144×10^{-2}	1.101
1	1.800×10^{-2}	1.018	7.310×10^{-2}	1.080
3	1.089×10^{-2}	1.011	4.675×10^{-2}	1.049
5	4.306×10^{-3}	1.004	2.084×10^{-2}	1.021
7	8.279×10^{-4}	1.001	5.585×10^{-3}	1.006
9	6.215×10^{-5}	1.000	7.139×10^{-4}	1.001

The expected number of transmissions as a function of z is shown on the following page.

(9.46) (Cont)



(b) The error performance is summarized

z dB	$P_E(\gamma=0.05)$	$P_E(\gamma=0.2)$	P_{EN}
-3	1.502×10^{-1}	1.271×10^{-1}	1.584×10^{-1}
-1	9.500×10^{-2}	7.177×10^{-2}	1.038×10^{-1}
1	4.872×10^{-2}	3.071×10^{-2}	5.628×10^{-2}
3	1.817×10^{-2}	8.667×10^{-3}	2.288×10^{-2}
5	4.156×10^{-3}	1.300×10^{-3}	5.954×10^{-3}
7	4.436×10^{-3}	7.300×10^{-5}	7.727×10^{-4}
9	1.426×10^{-5}	8.654×10^{-7}	3.364×10^{-5}

The greatest improvement comes for high signal-to-noise ratios since the variance is small. Note that for z = 9 dB the improvement is well over an order of magnitude with γ = 0.2. Reference to part (a) shows that this is accomplished negligible retransmissions.

(9.47)

From Chapter 6 (Equation (6.96)) the output signal-to-noise ratio for an FM system with preemphasis is

$$(SNR)_{DF} = \left(\frac{f_d}{f_3}\right)^2 \overline{m^2} \; \frac{P_T}{N_0 W}$$

Since the modulation index, β, is defined rather than the deviation ratio, D, sinusoidal modulation is implied. Thus $\overline{m^2} = 1/2$. Also $f_d = \beta W$. Thus

$$(SNR)_{DF} = \beta^2 \left(\frac{W}{f_3}\right)^2 \frac{1}{2} \frac{P_T}{N_0 W}$$

For commercial FM $\beta = 5$, $W = 15$ KHz and $f_3 = 2.1$ KHz. Thus we shall assume W and f_3 equal to these values and vary β. This yields

$$(SNR)_{DF} = \beta^2 \left(\frac{15}{2.1}\right)^2 \frac{1}{2} \frac{P_T}{N_0 W}$$

$$= 25.51 \beta^2 \frac{P_T}{N_0 W} .$$

Considering the optimal modulation system

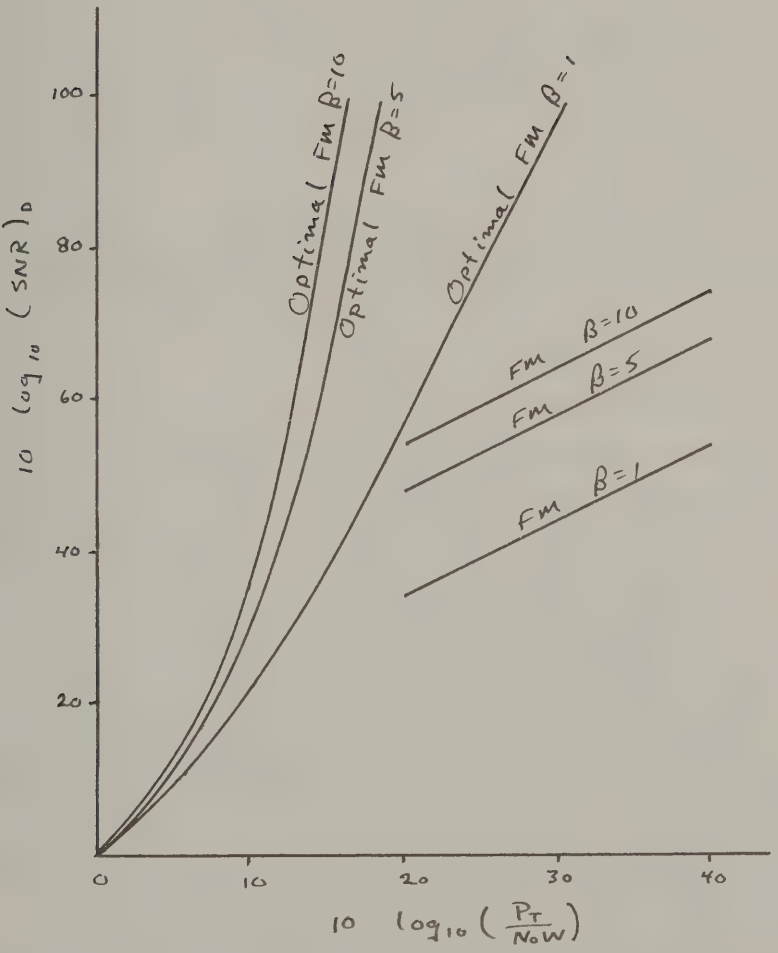
$$\gamma = \frac{B_T}{W} = \frac{2(\beta+1)W}{W} = 2(\beta+1)$$

and we shall let $\gamma = 4, 12$ and 22, which corresponds to $\beta = 1, 5$ and 10 respectively. For the optimal modulation system

(9.47) (Cont.)

$$(SNR)_D = \left[1 + \frac{1}{\gamma}\left(\frac{P_T}{N_0 W}\right)\right]^{\gamma} - 1 \ .$$

The plot follows.



(A.1)

$V_{rms} = \sqrt{4KTRB}$, $K = 1.38 \times 10^{-23}$ J/°K

$B = 10$ MHz $= 10^7$.

(a) $R = 10 \times 10^3\ \Omega$; $T = T_o = 290°$ K.

$V_{rms} = [(4)(1.38 \times 10^{-23})(290)(10^4)(10^7)]^{1/2}$

$= 4.0 \times 10^{-5}$ volts rms

$= 40.0\ \mu$volts.

(b) $R = 10^4\ \Omega$, $T = 29°$ K.

V_{rms}^2 is smaller than part (a) by a factor of 10.

∴ V_{rms} is smaller by $\sqrt{10} = 3.16$

$V_{rms} = 12.70\ \mu$volts.

(c) $R = 10^4\ \Omega$, $T = 2.9°$ K.

This result for V_{rms} will be smaller than part (b) by a factor of $\sqrt{10}$, so

$V_{rms} = (12.70 \times 10^{-6}) / 3.16 = 4.00\ \mu$volts

(d) Each answer becomes smaller by factors of 2, $\sqrt{10} = 3.16$, and 10 , respectively.

(A.2)

$I = I_s \left[\exp\left(\frac{eV}{KT}\right) - 1\right]$.

We want $I > 20\ I_s$ or $\exp\left(\frac{eV}{KT}\right) - 1 > 20$.

(a) At $T = 290°$ K , $\frac{e}{KT} \approx 40$, so we have

$\exp(40V) > 21 \Rightarrow V > \frac{1}{40} \ln(21) = .0761$ volts.

For $V = .0761$ volts and $I_s = 10\ \mu$A ,

$i_{rms}^2 \approx 2eIB \approx 2eBI_s \exp\left(\frac{eV}{KT}\right)$

(A.2) (Cont.)

Thus

$$i_{rms}^2/B = 2e I_s \exp\left(\frac{eV}{KT}\right) = 2(1.6\times10^{-19})(10^{-5})\exp[40(.076)]$$
$$= 6.72\times10^{-23} \text{ Amp}^2/\text{Hz}.$$

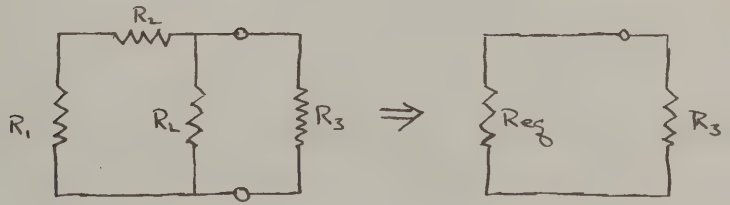
(b) If $T = 29°K$, then $\frac{e}{KT} = 400$ and for $I > 20\,I_s$ we need $\exp(400V) > 21$ or $V > \frac{1}{400}\ln 21 = 7.61\times10^{-3}$.

Thus

$$i_{rms}^2/B = 3.2\times10^{-24}\exp[400(7.61\times10^{-3})] = 6.72\times10^{-23} \text{ Amp}^2/\text{Hz}.$$

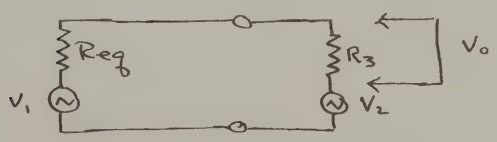
(A.3)

We can use summation of independent noise powers or Nyquists' formula. To use Nyquists' formula, we consider the equivalent circuit



where $R_{eq} = R_L \| (R_1 + R_2) = \dfrac{R_L(R_1+R_2)}{R_1+R_2+R_L}$.

The noise equivalent circuit is



A.3 (Cont.)

where

$$V_1^2 = 4KTR_{eq}B$$

$$V_2^2 = 4KTR_3B$$

$$V_o^2 = \left(\frac{R_3}{R_{eq}+R_3}V_1\right)^2 + \left(\frac{R_3}{R_{eq}+R_3}V_2\right)^2.$$

The expression for V_o^2 can be reduced as follows.

$$V_o^2 = \frac{R_3^2}{(R_3+R_{eq})^2}(V_1^2+V_2^2) = \frac{4KTB(R_{eq}+R_3)R_3^2}{(R_3+R_{eq})^2}$$

$$= \frac{4KTBR_3^2}{R_3+R_{eq}}$$

(b) From part (a)

$$\frac{V_o^2}{B} = \frac{4KTR_3^2}{R_3+R_{eq}} \quad \text{where } R_{eq} = \frac{R_L(R_1+R_2)}{R_1+R_2+R_L}$$

$$R_{eq} = \frac{500(1000+500)}{1000+500+500} = 375.$$

Therefore

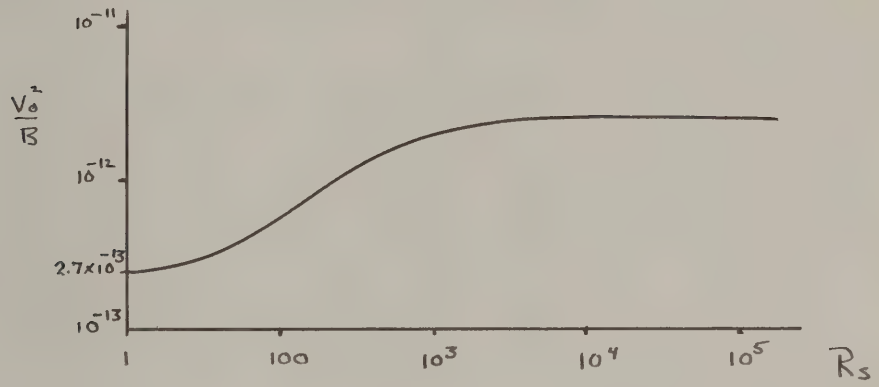
$$\frac{V_o^2}{B} = \frac{4(1.38\times10^{-23})(290)(10)^2}{10+375} = 4.16\times10^{-21}\ V^2/Hz.$$

A.4

It turns out that the minimum for V_o^2/B occurs at $R_s=0$ as may be verified by putting in typical values for the parameters. Using the values given in Ex. A.2,

(A.4) (Cont.)

the following plot of $\overline{v_o^2}/B$ as a function of R_s is obtained.

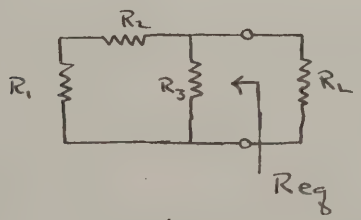


Letting $R_s = 0$,

$$\frac{\overline{v_o^2}}{B} = 4KTR_L\left[1+\frac{eI_cR_L}{2KT}+\frac{g_m^2R_L r_\pi^2 r_b}{(r_\pi+r_b)^2}\left(1+\frac{r_b}{2r_\pi}\right)\right].$$

For values of Ex. A.2, $\frac{\overline{v_o^2}}{B} = 2.74\times10^{-13}$ which is about a factor of 6 smaller.

(A.5)



$$R_{eq} = R_3 \parallel (R_1+R_2) = \frac{R_3(R_1+R_2)}{R_1+R_2+R_3}$$

$$\therefore \quad R_L = \frac{R_3(R_1+R_2)}{R_1+R_2+R_3}.$$

(A.6)

(a) $$F = 1 + \frac{r_b}{R_s}\left[1 + \frac{(r_b + R_s)^2}{2 r_\pi r_b} + \left(\frac{r_\pi + r_b + R_s}{g_m r_\pi}\right)^2 \frac{e I_c}{2KT_o r_b}\right]$$

Note that

$$g_m = \frac{e I_c}{KT_o} \quad ; \quad r_\pi = \frac{KT_o}{e I_B}$$

Therefore

$$F = 1 + \frac{r_b}{R_s}\left[1 + \frac{(r_b + R_s)^2}{2 r_\pi r_b} + \frac{1}{g_m r_\pi}\frac{(r_\pi + r_b + R_s)^2}{2 r_\pi r_b}\right].$$

$$\frac{dF}{dR_s} = -\frac{r_b}{R_s^2}\left[1 + \frac{(r_b+R_s)^2}{2 r_\pi r_b} + \frac{1}{g_m r_\pi}\frac{(r_\pi + r_b + R_s)^2}{2 r_\pi r_b}\right]$$

$$+ \frac{r_b}{R_s}\left[\frac{2(r_b + R_s)}{2 r_\pi r_b} + \frac{1}{g_m r_\pi}\frac{2(r_\pi + r_b + R_s)}{2 r_\pi r_b}\right].$$

Multiplying thru by $2 r_\pi R_s^2$ and simplifying gives

$$\frac{dF}{dR_s} = -\left[2 r_\pi r_b + (r_b + R_s)^2 + \frac{(r_\pi + r_b + R_s)^2}{g_m r_\pi}\right]$$

$$+ R_s\left[2(r_b + R_s) + \frac{2}{g_m r_\pi}(r_\pi + r_b + R_s)\right] = 0$$

Solution gives

$$R_s = \left[\frac{2 r_\pi^2 r_b g_m + (r_\pi + r_b)^2 + g_m r_\pi r_b^2}{1 + g_m r_\pi}\right]^{1/2}.$$

Note that the value for R_s depends only on parameters pertaining to the transistor.

(b), (c)

The values of Ex. A.3 are

$g_m = .04$ mho ; $r_b = 100\,\Omega$; $R_L = 10\,K\Omega$; $r_\pi = 2500\,\Omega$.

$$R_{s,opt.} = \left[\frac{2(2500)(100) + \frac{(2500+100)^2}{.04(2500)} + (100)^2}{1 + 1/[.04(2500)]}\right]^{1/2} = 756.23\,\Omega.$$

A.6 (Cont.)

$$F_{min} = 1 + \frac{110}{756.2}\left[1 + \frac{(100+756.2)^2}{2(2500)(100)} + \frac{(2500+100+756.2)^2}{100(.04)(2)(2500)(100)}\right]$$

$$= 2.071$$

(d) $r_\pi = \frac{KT}{eI_B} = \frac{KT\beta}{eI_c}$; $I_c = \beta I_B$; $g_m = \frac{eI_c}{KT}$

$r_\pi g_m = \beta$; $\frac{eKT}{KT_0} = 40\ v^{-1}$.

$$F = 1 + \frac{r_b}{R_s}\left\{1 + \frac{eI_c}{2KT\beta r_b}\left[(r_b+R_s)^2 + \frac{1}{\beta}\left(\frac{KT\beta}{eI_c} + r_b + R_s\right)^2\right]\right\}.$$

$$\frac{dF}{dI_c} = (r_b+R_s)^2 + \frac{1}{\beta}\left(\frac{KT\beta}{eI_c} + r_b + R_s\right)^2 - \frac{2KT}{eI_c}\left(\frac{KT\beta}{eI_c} + r_b + R_s\right) = 0$$

Solution gives

$$\frac{KT}{eI_{c,opt}} = (r_b+R_s)\frac{\sqrt{1+\beta}}{\beta} \approx \frac{r_b+R_s}{\sqrt{\beta}}$$

$$\therefore I_{c,opt.} = \frac{KT}{e}\frac{\beta}{(r_b+R_s)\sqrt{1+\beta}}.$$

(e)

Using the previous results, the expression for F_{min} becomes

$$F_{min} = 1 + \frac{r_b}{R_s}\left\{1 + \frac{1}{2\sqrt{\beta}\, r_b}\left[(r_b+R_s) + \frac{1}{\beta}(r_b+R_s)(\sqrt{\beta}+1)^2\right]\right\}$$

$$= 1 + \frac{r_b}{R_s}\left\{1 + \frac{(r_b+R_s)}{2\sqrt{\beta}\, r_b}\left[1 + \underbrace{\frac{(\sqrt{\beta}+1)^2}{\beta}}\right]\right\}$$

≈ 1 for $\beta \gg 1$

$$\approx 1 + \frac{r_b}{R_s}\left\{1 + \frac{r_b+R_s}{\sqrt{\beta}\, r_b}\right\},\ \beta \gg 1$$

$$\approx 1 + \frac{r_b}{R_s} + \frac{1}{\sqrt{\beta}},\ r_b \ll R_s.$$

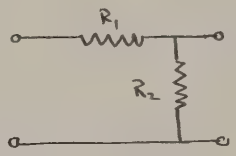
Therefore

$$F_{min} = 1 + \frac{100}{750}\left(1 + \frac{100+750}{10(100}\right)$$

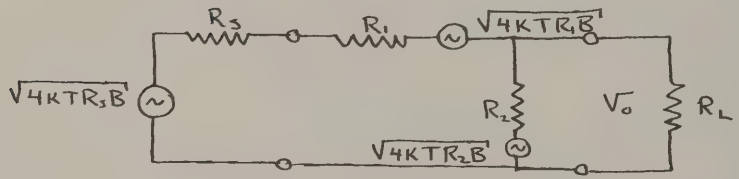
$$= 1.25.$$

A.7

Want to find the noise figure of the network shown below.



The technique is to find the noise power due to source and due to source and network. First assume system is matched.



$$\overline{V_o^2}\Big|_{\text{due to } R_s \text{ only}} = \left(\frac{R_2 \| R_L}{R_1 + R_s + R_2 \| R_L}\right)^2 4KTR_sB$$

$$\overline{V_o^2}\Big|_{\text{due to } R_1 \& R_2} = \left(\frac{R_2 \| R_L}{R_1 + R_s + R_2 \| R_L}\right)^2 4KTR_1B + \left(\frac{R_L \| (R_1 + R_s)}{R_2 + (R_1 + R_s) \| R_L}\right)^2 4KTR_2B$$

Therefore

$$\overline{V_o^2}\Big|_{\text{due to } R_s, R_1, R_2} = \left(\frac{R_2 \| R_L}{R_1 + R_s + R_2 \| R_L}\right)^2 (4KTR_sB + 4KTR_1B) + \left(\frac{R_L \| (R_1 + R_s)}{R_2 + (R_1 + R_s) \| R_L}\right)^2 4KTR_2B$$

and the noise figure is

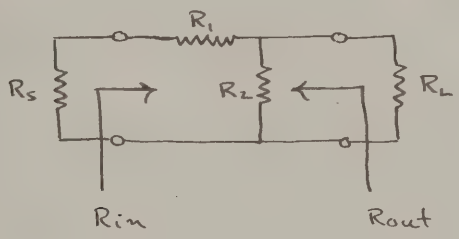
(A.7) (Cont.)

$$F = 1 + \frac{R_1}{R_S} + \left[\frac{R_L \parallel (R_1+R_S)}{R_2+(R_1+R_S)\parallel R_L}\right]^2 \left[\frac{R_S+R_1+R_2\parallel R_L}{R_2\parallel R_L}\right]^2 \frac{R_2}{R_S} .$$

So, for general R_S and R_L we have

$$F = 1 + \frac{R_1}{R_S} + \left[\frac{(R_1+R_S)(R_2R_S+R_2R_S+R_LR_S+R_LR_1)}{(R_1R_L+R_2R_S+R_2R_L+R_1R_L+R_SR_L)R_2}\right]^2 \frac{R_2}{R_S} .$$

Note that the noise in R_L has been excluded. This belongs to next stage. Now this is a matching circuit and so we want input matched and output matched.



$$R_S = R_{in} = R_1 + R_2 \parallel R_L \quad ; \quad R_L = R_{out} = R_2 \parallel (R_1+R_S)$$

So

$$R_S = R_1 + \frac{R_2R_L}{R_2+R_L} \quad ; \quad R_L = \frac{R_2(R_1+R_S)}{R_1+R_2+R_S} .$$

Next these expressions are substituted back into the expression for F given above. This gives (after simplification)

$$F = 1 + \frac{R_1}{R_S} + \left[\frac{2R_L^2R_S(R_1+R_2+R_S)/(R_S-R_1)}{R_2^2(R_1+R_S+R_L)+R_L^2(R_1+R_2+R_S)}\right]^2 \frac{R_2}{R_S}$$

A.7 (Cont.)

Note that if $R_1 \gg R_2$, we then have matched condition if $R_L = R_2$ and $R_S = R_1$.

Then

$$F = 1 + 1 + \left[\frac{1 \cdot \frac{(2R_1)R_2}{2R_1+R_2} \cdot \left(2R_1 + \frac{R_2^2}{R_2}\right)}{\left(R_2 + \frac{2R_1R_2}{2R_1+R_2}\right)\left(\frac{R_2^2}{2R_2}\right)} \right]^2 \frac{R_2}{R_1} .$$

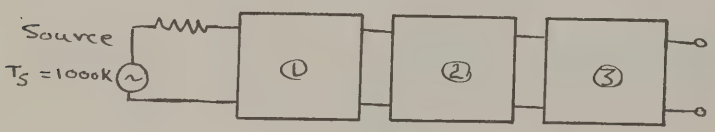
This expression simplifies to give $F = 2 + \frac{16R_1}{R_2}$

Note: The "cheap" matching resistive pad is very poor from a noise figure point of view.

To find T_e, we use the relation

$F = 1 + \frac{T_e}{T_o} \Rightarrow T_e = T_o(F-1)$.

A.8



$F_\ell = 1 + \frac{T_e}{T_o}$; $T_e = T_o(F-1)$.

$T_e = T_{e1} + T_{e2}/G_{a1} + T_{e3}/G_{a1}G_{a2} + \ldots$

Amp #	F	T_{ei}	G_{ai}
1		300°K	10dB = 10
2	10dB = 10	2610K	30dB = 1000
3	17dB = 50.1	14529K	30dB = 1000

Therefore

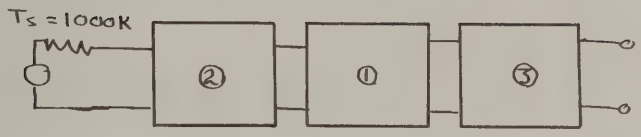
$$T_{eo} = 300 + \frac{2610}{10} + \frac{14529}{10(1000)} = 562.45$$

(A.8) (Cont.)

and so

$$F = 1 + \frac{T_{eo}}{T_o} = 1 + \frac{562.45}{290} = 2.94 = 4.68\,dB$$

(b)



$$T_{eo} = 2610 + \frac{300}{1000} + \frac{14529}{10(1000)}$$

$$= 2611.76$$

and

$$F = 1 + \frac{2611.76}{290} = 10.01$$

(c) See parts (a), (b).

(d)

$B = 100\,KHz$, $T_s = 1000°K$, $G_a = 10^7$. For part (a):

$$P_{na,out} = G_a k (T_s + T_{eo}) B$$

$$= 10^7 (1.38 \times 10^{-23})(1562.45)(10^5) = 2.156 \times 10^{-8}\ W.$$

We want

$$\frac{P_{sa,out}}{P_{na,out}} = 10^4 = \frac{P_{sa,out}}{2.156 \times 10^{-8}}$$

Thus

$$P_{sa,out} = 2.156 \times 10^{-4} \text{ and } P_{sa,in} = 2.156 \times 10^{-11}\ W.$$

For part (b):

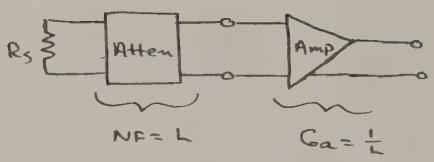
$$P_{na,out} = 10^7 (1.38 \times 10^{-23})(1000 + 2611.76)(10^5) = 4.984 \times 10^{-8}.$$

and so $P_{sa,out} = 4.984 \times 10^{-4} \Rightarrow P_{sa,in} = 4.984 \times 10^{-11}\ W.$

Note small change from first result due to large temperature of source.

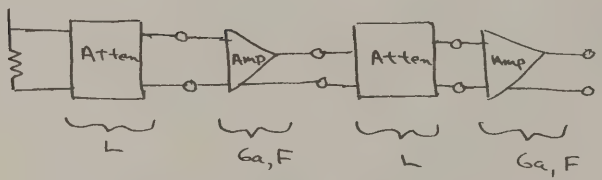
A.9

(a)



$$F_{overall} = F_1 + \frac{F_2 - 1}{G_{a_1}} = L + \frac{F-1}{1/L} = L + L(F-1) = LF$$

(b)



$$F_{overall} = L + \frac{F-1}{1/L} + \frac{L-1}{\frac{1}{L}G_a} + \frac{F-1}{\frac{1}{L}G_a\frac{1}{L}}$$

$$= L + L(F-1) + (L-1) + L(F-1)$$

$$\approx LF + LF = 2LF.$$

(c) Generalizing,

$$F_{overall} \approx NLF.$$

A.10

(a) Free-Space Loss $= 20\log_{10}[(.03896)/(4\pi \times 41 \times 10^6)]$

$= -202.43$ dB

EIRP $=$ 74.18 dBW

$G_{R,dB} =$ 6.0 dB

Losses $=$ −5.0 dB

∴ Received Signal Level $= -127.25$ dBW.

(A.10) (Cont.)

(b) $P_{int,\,dBm} = 10\log_{10}\left[kT_0\left(\frac{T_e}{T_0}\right)B\right]$

$= 10\log_{10} kT_0 + 10\log_{10}\frac{T_e}{T_0} + 10\log_{10} B$

$= -174 + 5.38 + 60$

$= -108.62\ dBm = -138.62\ dBW.$

(c) $(SNR)_{Rec} = (-127.25) - (-138.62)$

$= 11.37\ dB.$

PART III

THE CALCULATION OF UNCODED AND CODED WORD ERROR PROBABILITIES

THE CALCULATION OF UNCODED AND CODED WORD WORD ERROR PROBABILITIES

The purpose of this section is to illustrate the calculation of coded and uncoded word error probabilities for block coded systems. An efficient FORTRAN IV computer program is included which can be used to solve problems 9.30 and 9.34 and accurately construct performance characteristics, such as Figure 9.14, for any desired range of signal-to-noise ratios.

Theory

The word error probability for an uncoded system is given by (9.49) as

$$P_{eu} = 1 - (1 - q_u)^k \qquad [1]$$

where

$$q_u = \frac{1}{2} \operatorname{erfc}\left[\sqrt{\frac{ST_w}{kN_o}}\right] \qquad [2]$$

is the uncoded symbol error probability. In the above expressions

S = received signal power (watts)

T_w = word duration (sec)

k = information symbols per word

N_o = single-sided noise power spectral density

Equation [1] is the probability of one or more symbol errors per word. Thus it can be written

$$P_{eu} = \sum_{i=1}^{k} \binom{k}{i} q_u^{\,i} (1 - q_u)^{k-i} \qquad [3]$$

This is the form which will be used for computation.

The word error probability for the coded system is given by (9.50) as

$$P_{ec} = \sum_{i=e+1}^{n} \binom{n}{i} q_c^{\,i} (1 - q_c)^{n-i} \qquad [4]$$

where

$$q_c = \frac{1}{2} \operatorname{erfc} \left[\sqrt{\frac{ST_w}{nN_o}} \right] \qquad [5]$$

is the symbol error probability for the coded system. In equations [4] and [5]

e = number of correctable errors per codeword

n = total number of symbols per codeword

and all other parameters are as defined previously.

The Computer Program

The program computes the uncoded and coded word error probabilities, [3] and [4], respectively, as a function of ST_w/N_o. There are seven input parameters

N = number of symbols per codeword

K = number of information symbols per codeword

NEC = number of correctable errors per codeword = e in [4]

RHO = This parameter is R_{12} in Equation (7.42), page 315. This parameter is set equal to 0 for FSK and ASK systems and -1 for PRK systems.

DBLL = The lower limit of ST_w/N_o in decibels

DBUL = The upper limit of ST_w/N_o in decibels

DBINC = Incremental value of ST_w/N_o in decibels

The main program simply consists of seven statements specifying these seven values followed by

```
CALL PERROR(N,K,NEC,RHO,DBLL,DBUL,DBINC)
```

The program runs until the specified upper limit of ST_w/N_o is reached or until a calculated error probability of 10^{-69} is reached.

The program is given on the following five pages.

```
      SUBROUTINE PERROR(N,K,NEC,ARHO,ADBLL,ADBUL,ADBINC)
C
C*************************************************************************
C*********************  INPUT PARAMETER DEFINITIONS  *********************
C*************************************************************************
C*                                                                       *
C*     N       - NUMBER OF SYMBOLS PER CODEWORD                          *
C*     K       - NUMBER OF INFORMATION SYMBOLS                           *
C*     NEC     - NUMBER OF CORRECTABLE ERRORS PER CODEWORD               *
C*     ARHO    - R12 (REFERENCE EQUATION 7.42)                           *
C*     ADBLL   - LOWER LIMIT IN DB OF (STW/NO)                           *
C*     ADBUL   - UPPER LIMIT IN DB OF (STW/NO)                           *
C*     ADBINC  - INCREMENTAL VALUE IN DB OF (STW/NO)                     *
C*                                                                       *
C*************************************************************************
C*                                                                       *
C*     CALL ERRSET(208,260,-1,1) - IBM SUBROUTINE TO SUPPRESS PRINTING   *
C*          OF UNDERFLOW ERROR MESSAGE AND TRACEBACK                     *
C*                                                                       *
C*************************************************************************
C
      CALL ERRSET(208,260,-1,1)
C
C*************************************************************************
C***************************  INITIALIZATION  ****************************
C*************************************************************************
C
C     >>>>>  CONVERT ALL PARAMETERS TO DOUBLE PRECISION  <<<<<
C
      IMPLICIT REAL*8(B-H,O-Z)
      ONE=1.0D0
      TWO=2.0D0
      TEN=10.0D0
      DBLL=DBLE(ADBLL)
```

```
      DBINC=DBLE(ADBINC)
      RHO=DBLE(ARHO)
C
C     >>>>>  COMPUTE NUMBER OF ITERATIONS  <<<<<
C
      L=IFIX(((ADBUL-ADBLL)/ADBINC)+1.999)
C
C*************************************************************************
C*****************  INITIALIZE UNCODED CASE PARAMETERS  ******************
C*************************************************************************
C
      WRITE(6,100)N,K,NEC
      WRITE(6,101)
      DB=DBLL-DBINC
      Z1=(ONE-RHO)/(TWO*DFLOAT(K))
      KM=K-1
C
C*************************************************************************
C**********  COMPUTE AND PRINT QU AND PEU FOR DESIRED VALUES  **********
C*************************************************************************
C
      DO 4 J1=1,L
      DB=DB+DBINC
      PEU=0.0D0
C
C     >>>>>  COMPUTE (STW/NO) FROM DB SCALE  <<<<<
C
      Z2=TEN**(DB/TEN)
      Z=Z1*Z2
C
C     >>>>>  COMPUTE QU  <<<<<
C
      QU=(0.5D0)*DERFC(DSQRT(Z))
      IF(QU.LT.1.0D-69)GO TO 5
```

```
C
C      >>>>>  COMPUTE PEU  <<<<<
C
       IF(K.EQ.1)GO TO 3
       DO 2 I=1,KM
       K1=K-I
       P=ONE
       QPU=(ONE-QU)*(QU**(DFLOAT(I)/DFLOAT(K1)))
       DO 1 J3=1,K1
     1 P=(DFLOAT(I+J3)/DFLOAT(J3))*QPU*P
       PEU=PEU+P
     2 CONTINUE
     3 PEU=PEU+(QU**K)
       WRITE(6,103)DB,QU,PEU
       IF(PEU.LT.1.0D-69)GO TO 5
     4 CONTINUE
     5 WRITE(6,104)
C
C*************************************************************************
C****************** INITIALIZE CODED CASE PARAMETERS  ********************
C*************************************************************************
C
       WRITE(6,100)N,K,NEC
       WRITE(6,102)
       DB=DBLL-DBINC
       Z1=(ONE-RHO)/(TWO*DFLOAT(N))
       NEC1=NEC+1
       NM=N-1
C
C*************************************************************************
C**********  COMPUTE AND PRINT QC AND PEC FOR DESIRED VALUES  ************
C*************************************************************************
C
       DO 9 J1=1,L
```

```
      DB=DB+DBINC
      PEC=0.0D0
C
C     >>>>>  COMPUTE (STW/NO) FROM DB SCALE  <<<<<
C
      Z2=TEN**(DB/TEN)
      Z=Z1*Z2
C
C     >>>>>  COMPUTE QC  <<<<<
C
      QC=(0.5D0)*DERFC(DSQRT(Z))
      IF(QC.LT.1.0D-69)GO TO 10
C
C     >>>>>  COMPUTE PEC  <<<<<
C
      IF(N.EQ.1)GO TO 8
      DO 7 I=NEC1,NM
      N1=N-I
      P=ONE
      QPC=(ONE-QC)*(QC**(DFLOAT(I)/DFLOAT(N1)))
      DO 6 J3=1,N1
    6 P=(DFLOAT(I+J3)/DFLOAT(J3))*QPC*P
      PEC=PEC+P
    7 CONTINUE
    8 PEC=PEC+(QC**N)
      WRITE(6,103)DB,QC,PEC
      IF(PEC.LT.1.0D-69)GO TO 10
    9 CONTINUE
   10 WRITE(6,104)
C
C*************************************************************************
C************************  FORMAT STATEMENTS  ****************************
C*************************************************************************
C
```

```
100 FORMAT('1 **********  N=',I4,5X,'K=',I4,5X,'F=',I4,'  **********')
101 FORMAT('- ******************  UNCODED CASE  ******************',/,
   1'  *',50X,'*',/,'  *',50X,'*',/,'  *  10LOG(STW/NO)',9X,'QU',13X,'
   2PEU',8X,'*',/,'  *',50X,'*')
102 FORMAT('- ******************  CODED CASE  ******************',/,
   1'  *',50X,'*',/,'  *',50X,'*',/,'  *  10LOG(STW/NO)',9X,'QC',13X,'
   2PEC',8X,'*',/,'  *',50X,'*')
103 FORMAT('  *   ',D11.4,5X,D11.4,5X,D11.4,4X,'*')
104 FORMAT('  **************************************************')
    RETURN
    END
```

An Example

As an example of the program, the performance of a (7,4) single error correcting Hamming code will be investigated. We shall assume phase reversal keying (PRK). Note that this gives Example 9.12. For this case the main program is

```
DBLL = 0.0
DBUL = 20.0
DBINC = 0.5
RHO = -1.0
N = 7
K = 4
NEC = 1
CALL PERROR(N,K,NEC,RHO,DBLL,DBUL,DBINC)
```

The computer output is given on the following two pages. This is followed by the performance curves for ST_w/N_o between 6 and 16 dB. Also given is the approximation to the coded word error probability.

$$P_{ec} \cong 21\, q_c^{\,2} = \frac{21}{4}\left[\operatorname{erfc}\left[\sqrt{\frac{ST_w}{7N_o}}\right]\right]^2$$

which is (9.56). It can be seen that the approximation is valid for values of ST_w/N_o greater than 11 dB.

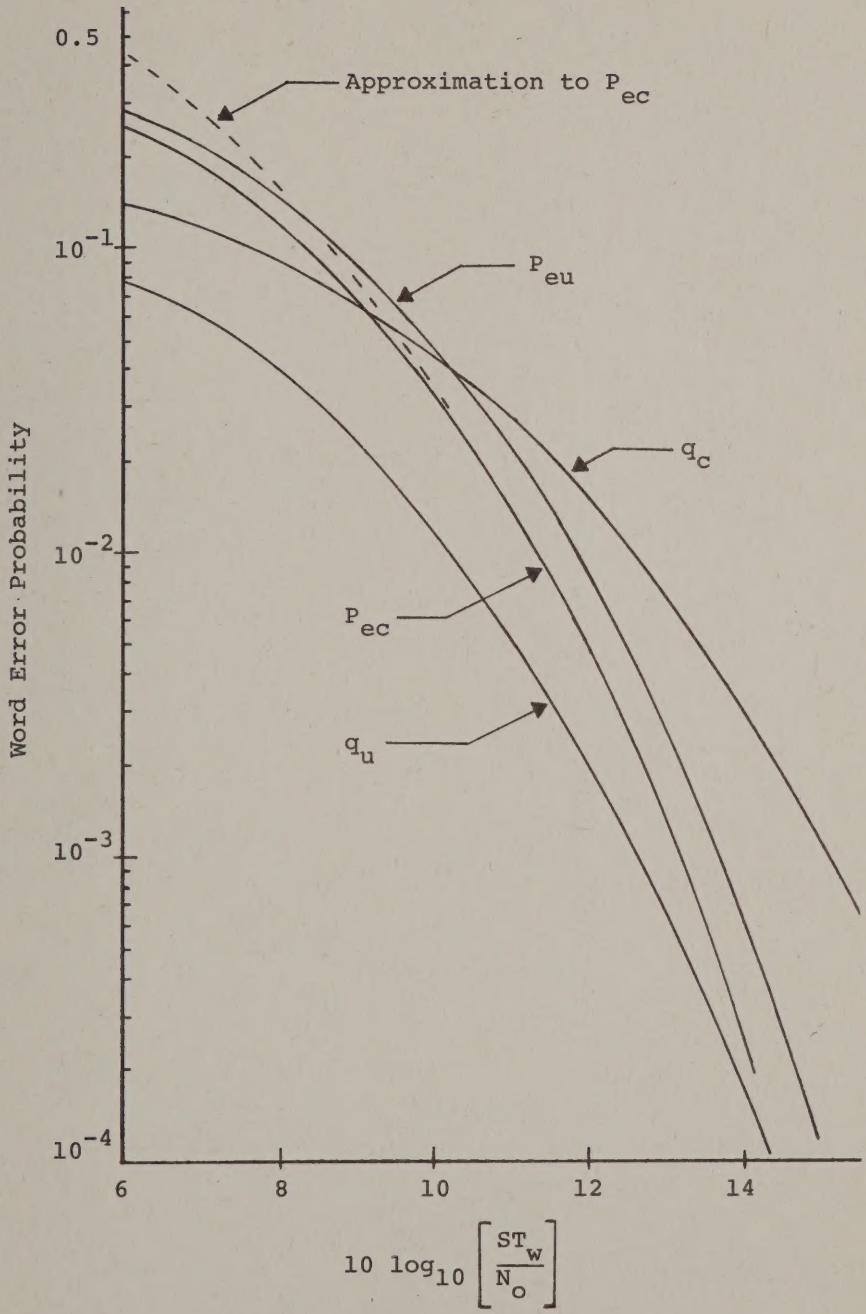
********** N= 7 K= 4 E= 1 **********

****************** UNCODED CASE ******************

10LOG(STW/NO)	QU	PEU
0.0	0.2398D+00	0.6659D+00
0.5000D+00	0.2269D+00	0.6428D+00
0.1000D+01	0.2138D+00	0.6179D+00
0.1500D+01	0.2003D+00	0.5911D+00
0.2000D+01	0.1867D+00	0.5624D+00
0.2500D+01	0.1729D+00	0.5319D+00
0.3000D+01	0.1589D+00	0.4996D+00
0.3500D+01	0.1450D+00	0.4657D+00
0.4000D+01	0.1312D+00	0.4303D+00
0.4500D+01	0.1176D+00	0.3937D+00
0.5000D+01	0.1043D+00	0.3563D+00
0.5500D+01	0.9144D-01	0.3186D+00
0.6000D+01	0.7914D-01	0.2809D+00
0.6500D+01	0.6753D-01	0.2440D+00
0.7000D+01	0.5671D-01	0.2083D+00
0.7500D+01	0.4679D-01	0.1744D+00
0.8000D+01	0.3785D-01	0.1430D+00
0.8500D+01	0.2996D-01	0.1146D+00
0.9000D+01	0.2314D-01	0.8938D-01
0.9500D+01	0.1739D-01	0.6775D-01
0.1000D+02	0.1267D-01	0.4974D-01
0.1050D+02	0.8929D-02	0.3524D-01
0.1100D+02	0.6055D-02	0.2400D-01
0.1150D+02	0.3935D-02	0.1565D-01
0.1200D+02	0.2439D-02	0.9719D-02
0.1250D+02	0.1433D-02	0.5718D-02
0.1300D+02	0.7928D-03	0.3168D-02
0.1350D+02	0.4104D-03	0.1641D-02
0.1400D+02	0.1971D-03	0.7882D-03
0.1450D+02	0.8705D-04	0.3481D-03
0.1500D+02	0.3499D-04	0.1400D-03
0.1550D+02	0.1266D-04	0.5063D-04
0.1600D+02	0.4069D-05	0.1627D-04
0.1650D+02	0.1145D-05	0.4582D-05
0.1700D+02	0.2780D-06	0.1112D-05
0.1750D+02	0.5710D-07	0.2284D-06
0.1800D+02	0.9729D-08	0.3892D-07
0.1850D+02	0.1344D-08	0.5377D-08
0.1900D+02	0.1468D-09	0.5873D-09
0.1950D+02	0.1232D-10	0.4927D-10
0.2000D+02	0.7687D-12	0.3075D-11

********** N= 7 K= 4 E= 1 **********

CODED CASE		
10LOG(STW/NO)	QC	PEC
0.0	0.2965D+00	0.6631D+00
0.5000D+00	0.2856D+00	0.6393D+00
0.1000D+01	0.2743D+00	0.6136D+00
0.1500D+01	0.2626D+00	0.5860D+00
0.2000D+01	0.2505D+00	0.5563D+00
0.2500D+01	0.2380D+00	0.5247D+00
0.3000D+01	0.2251D+00	0.4911D+00
0.3500D+01	0.2119D+00	0.4558D+00
0.4000D+01	0.1985D+00	0.4190D+00
0.4500D+01	0.1848D+00	0.3810D+00
0.5000D+01	0.1709D+00	0.3422D+00
0.5500D+01	0.1570D+00	0.3030D+00
0.6000D+01	0.1431D+00	0.2642D+00
0.6500D+01	0.1293D+00	0.2262D+00
0.7000D+01	0.1157D+00	0.1899D+00
0.7500D+01	0.1025D+00	0.1559D+00
0.8000D+01	0.8969D-01	0.1248D+00
0.8500D+01	0.7748D-01	0.9707D-01
0.9000D+01	0.6597D-01	0.7318D-01
0.9500D+01	0.5527D-01	0.5327D-01
0.1000D+02	0.4548D-01	0.3729D-01
0.1050D+02	0.3669D-01	0.2500D-01
0.1100D+02	0.2894D-01	0.1597D-01
0.1150D+02	0.2227D-01	0.9669D-02
0.1200D+02	0.1667D-01	0.5519D-02
0.1250D+02	0.1210D-01	0.2951D-02
0.1300D+02	0.8479D-02	0.1468D-02
0.1350D+02	0.5718D-02	0.6736D-03
0.1400D+02	0.3693D-02	0.2828D-03
0.1450D+02	0.2272D-02	0.1076D-03
0.1500D+02	0.1324D-02	0.3666D-04
0.1550D+02	0.7264D-03	0.1105D-04
0.1600D+02	0.3723D-03	0.2907D-05
0.1650D+02	0.1768D-03	0.6563D-06
0.1700D+02	0.7712D-04	0.1249D-06
0.1750D+02	0.3057D-04	0.1963D-07
0.1800D+02	0.1089D-04	0.2489D-08
0.1850D+02	0.3439D-05	0.2483D-09
0.1900D+02	0.9493D-06	0.1892D-10
0.1950D+02	0.2253D-06	0.1066D-11
0.2000D+02	0.4515D-07	0.4281D-13



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